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LARRY D. NACE

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by

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Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
ELECTRICAL ENGINEERING

United States Naval Postgraduate School
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the thesis requirements for the degree of

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ABSTRACT

The application of Mitrovic's Method enables the control systems engineer to analyze and synthesize control systems as a function of two parameters. The feedback compensation of sampled-data control systems is thoroughly investigated as a multi-parameter problem, with emphasis placed on the design of the compensation in the Mitrovic parameter plane. The minimization of the control system settling time is shown to be particularly adapted to the parameter plane. Several examples are included which demonstrates this usefulness of the parameter plane in solving the minimum settling time problem. A stability relationship is devised for digital computer determination of dynamic system stability which permits the formulation of an n-dimensioned parameter space. Applications of this digital creation of an n-dimensioned space are considered. This ability to design feedback compensation for the sampled-data system as a function of more than one parameter is shown to lead to greater accuracy in closed loop root positioning; and in certain cases, to permit the design consideration of several constraints simultaneously.

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TABLE OF SYMBOLS AND ABBREVIATIONS

Symbol	Definition
$\underline{A}$	A column vector whose components are $A_1, A_2, \ldots A_n$
a_k	Coefficient of the k-th ordered variable in a given polynomial.
D(s)	s-plane denominator polynomial of a transfer function.
G(s)	s-plane transfer function for the forward path, plant, of a feedback control system.
H(s)	s-plane transfer function for the feedback path of a feedback control system.
K	Root locus gain number for the transfer function G(s).
M-point	Any specified operating point on the Mitrovic Curves.
nT	The n-th sampling interval of a system whose sampling rate is 1/T cps.
p	The s-plane pole whose location is s = -p.
s	The complex LaPlace transform variable.
T	The interval between samples for a sampled-data system.
T_k	A Chebyshev function of the first kind.
t_s	settling time, the length or duration of the transient response for a stable system.
U_k	A Chebyshev function of the second kind.
z	The complex z-transform variable.
Z	The s-plane zero whose location is s = -Z.
Z.O.H.	A zero order hold circuit.
λ_i	the i-th eigenvalue of a given matrix.
ζ_s	the cosine of the angle θ , measured clockwise from the negative real axis of the s-plane.
ζ_z	The cosine of the angle ϕ , measured counter clockwise from the positive real axis of the z-plane.

Symbol	Definition
σ_Z	A line in the z-plane with constant real part.
ω_N	A circle of constant magnitude radius in the s-plane.
ω_Z	A circle of constant magnitude radius in the z-plane.

CHAPTER I

GENERAL CONCEPTS AND THEORY

1.1 Introduction.

The design of multivariable feedback compensation for a feedback control system, linear or sampled-data, has heretofore been accomplished largely by one of two methods. Either repeated calculations using the classical one parameter design techniques, such as root locus, has been used; or, if a more accurate study is required, the designer could resort to analog simulation and a "trial and error" approach to the design problem. Dusan Mitrovic, [3], in his early work with the algebraic methods for locating the roots of a polynomial introduced a new analysis technique that permitted analysis of the root location in terms of two variable coefficients. Later, Siljak, [4] and [5], devised additional algebraic methods whereby two parameters could be selected from a portion of any, or all, of the polynomial coefficients, and the root analysis performed in terms of these parameters. This extension has proven exceptionally useful in two parameter analysis of control systems whose characteristic equations have coefficients that are each functions of the same two variables or system parameters.

The object of this thesis, then is to apply the results of Mitrovic and Siljak to the design of feedback compensation for the sampled-data control system. A secondary objective is to show the superiority of the two parameter method over the more classical techniques, and to demonstrate the adaptability of the entire design problem to solution by digital computer.

1.2 Basic Equations.

Let us consider the characteristic equation of the sampled-data feedback control system as,

$$F(z) = 1 + GH(z) = 0 \tag{1a}$$

or,

$$F(z) = \sum_{K=0}^{n} a_k z^k = 0 \quad , \tag{1b}$$

where the a_k's are constant or parameter varying coefficients, and z is one of the complex z-plane locations of the closed loop roots of the system. Now, consider the generalized point in the z-plane,

$$z = \omega_z \zeta_z + j\omega_z(1 - \zeta_z^2)^{1/2} \tag{2}$$

where

$$|z| = \omega_z$$

and

$$\arg z = e^{j\phi}$$

$$= \cos\phi = j \sin\phi$$

$$= \zeta_z + j(1 - \zeta_z^2)^{1/2}$$

If this general coordinate is substituted into equation (1b), the characteristic equation may then be written as:

$$F(z) = \sum_{K=0}^{n} a_K \left[\omega_z \zeta_z + j\omega_z(1 - \zeta_z^2)^{1/2}\right]^K \tag{3}$$

Equation (3) is, in general, a non-zero quantity and assumes a zero value only when the value of ω_z and ζ_z assumed corresponds to that of one of the closed loop roots of the system. Thus, if we require that F(z) = 0, equation (3) describes the closed loop pole locations if the a_k's are constants, and every point on the z-plane root locus if one of the a_k's is parameter varying. Therefore,

$$\sum_{K=0}^{n} a_k(\omega_z \zeta_z + j\omega_z(1 - \zeta_z^2)^{1/2})^k = 0 \tag{4}$$

maps the characteristic equation from the z-plane to the Mitrovic plane if the a_k's are constants, and the parameter plane if the a_k's are variable.

Only the parameter plane will be discussed hereafter.

1.3 Conformal Mapping Techniques.

The requirement that equation (4) be identically equal to zero implies that both the real and imaginary parts are zero. Siljak, [5], demonstrated that this magnitude requirement on equation (4) leads to the two simultaneous equations,

$$\sum_{k=0}^{n} a_k \omega_z^k T_k(\zeta_z) = 0 \tag{5a}$$

and,

$$\sum_{k=0}^{n} a_k \omega_z^k U_k(\zeta_z) = 0 \quad , \tag{5b}$$

where $T_k(\zeta_z)$ and $U_k(\zeta_z)$ are the Chebyshev functions of the first and second kind respectively. Furthermore, since

$$T_k(\zeta_z) = \zeta_z U_k(\zeta_z) - U_{k-1}(\zeta_z) \quad , \tag{6}$$

equations (5a) and (5b) can be rewritten as

$$\sum_{k=0}^{n} - a_k \omega_z^k U_{k-1}(\zeta_z) = 0 \tag{7a}$$

and

$$\sum_{k=0}^{n} a_k \omega_z^k U_k(\zeta_z) = 0 \tag{7b}$$

respectively, with the $U_k(\zeta_z)$'s generated by the recursive Chebyshev formula

$$U_{k+1}(\zeta_z) = 2\zeta_z U_k(\zeta_z) - U_{k-1}(\zeta_z) \tag{8}$$

with

$$U_0(\zeta_z) \triangleq 0$$

$$U_1(\zeta_z) \triangleq 1$$

$$U_{-1}(\zeta_z) \triangleq -1$$

Siljak, [4] and [5], considered the case in which the a_k's are linear functions of two scalar parameters,

$$a_k = \alpha b_k + \beta c_k + d_k \quad . \tag{9}$$

The substitution of equation (9) into equations (7a) and (7b) forms the set of simultaneous equations,

$$\alpha B_1(\omega_z, \zeta_z) + \beta C_1(\omega_z, \zeta_z) + D_1(\omega_z, \zeta_z) = 0 \tag{10a}$$

and

$$\alpha B_2(\omega_z, \zeta_z) + \beta C_2(\omega_z, \zeta_z) + D_2(\omega_z, \zeta_z) = 0 \quad , \tag{10b}$$

where

$$B_1 = \sum_{K=0}^{m\le n} -b_k \omega_z^K U_{k-1}(\zeta_z) \quad , \quad B_2 = \sum_{K=0}^{m\le n} b_k \omega_z^K U_k(\zeta_z)$$

$$C_1 = \sum_{K=0}^{p\le n} -c_k \omega_z^K U_{k-1}(\zeta_z) \quad , \quad C_2 = \sum_{K=0}^{p\le n} c_k \omega_z^K U_k(\zeta_z)$$

$$D_1 = \sum_{K=0}^{q\le n} -d_k \omega_z^K U_{k-1}(\zeta_z) \quad , \quad D_2 = \sum_{K=0}^{q\le n} d_k \omega_z^K U_k(\zeta_z) \quad .$$

Equations (10a) and (10b) can then be solved simultaneously for the two parameters, alpha and beta as;

$$\alpha = \frac{C_1D_2 - C_2D_1}{B_1C_2 - B_2C_1} \quad , \quad \beta = \frac{B_2D_1 - B_1D_2}{B_1C_2 - B_2C_1} \quad . \tag{11}$$

Equations (11) are the fundamental results of Siljak, [4] and [5], and provide the basis for the work which follows. Equation (11) can be interpreted as the calculated values of the two parameters alpha and beta required to put a closed loop root at a particularly chosen location on the z-plane. Thus, by fixing ω_z and varying ζ_z from -1 to +1, upper semi-circle contours of constant ω_z can be mapped on the parameter plane. Thus, the argument of each upper semi-circle complex root conformally maps onto a single point on the parameter plane, the coordinates of which are the values of the two parameters producing that particular root configuration. Similiarly, in equation (11), ζ_z can be fixed while ω_z is varied from zero to infinity producing loci of constant ζ_z , or angle, on the parameter plane.

A third type of conformal mapping is yet required to completely specify all the roots of the characteristic equation on the parameter plane. The loci of constant ω_z and constant ζ_z determine all the complex roots. The loci of points on the parameter plane corresponding to constant real roots on the z-plane can be obtained directly from the characteristic equation, equation (1b). The substitution of equation (9) into equation (1b), and letting $z = \sigma_z$, yields

$$\alpha \sum_{K=0}^{m} b_k \sigma_z^K + \beta \sum_{K=0}^{P} c_k \sigma_z^K + \sum_{K=0}^{Q} d_k \sigma_z^K = 0 \quad , \qquad (12)$$

which for a given σ_z is a straight line on the parameter plane, regardless of the order of the characteristic equation.

CHAPTER II

SECOND ORDER SYSTEMS

The second order sampled-data control system provides an excellent model for developing the feedback compensation techniques from applications of the equations of Chapter I. For the initial design application, consider the pure second order system shown in figure (1a). The selected design goal is to minimize the settling time by suitable choice of the feedback parameters, a_1 and a_2. If the states of the system of figure (1a) are defined as:

$$X_1(t) \triangleq c(t)$$

and

$$X_2(t) \triangleq \dot{c}(t) \quad , \tag{13}$$

then the system can be represented by the signal flow-graph of figure (1b), and the vector-matrix equation

$$\underline{X}[(k+1)T] = \underline{\Phi}\ \underline{X}(kT) + \underline{\Delta}\ R(kT), \tag{14}$$

where

$$\underline{\Phi} = \begin{bmatrix} 1 - (a_1KT^2)/2 & T - (a_2KT^2)/2 \\ -(a_1KT) & 1 - (a_2KT) \end{bmatrix} \tag{15}$$

and

$$\underline{\Delta} = \begin{bmatrix} KT^2/2 \\ KT \end{bmatrix} \quad . \tag{16}$$

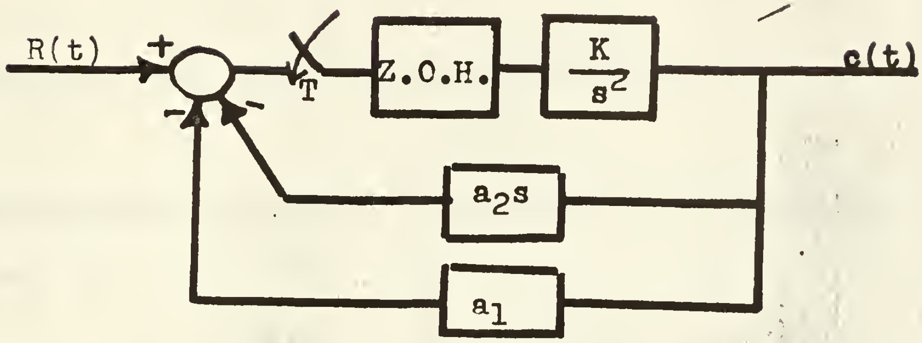
The characteristic equation of the system is;

$$2z^2 + z(2a_2KT + a_1KT^2 - 4) + (2 - 2a_2KT + a_1KT^2) = 0$$

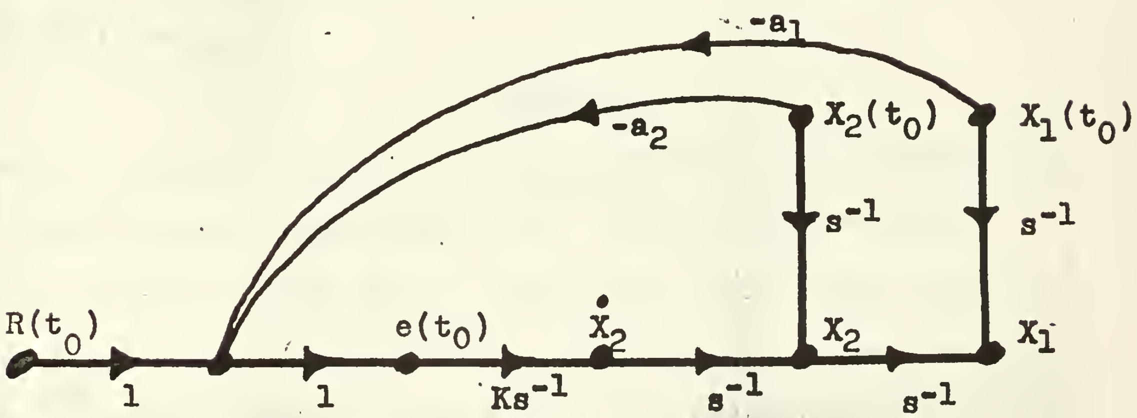
2.1 Constant Magnitude Loci.

The duration of the transient response, or settling time for a linear system of the second order is, [6],

Figure 1
Pure Second Order System
Diagrams



(a) Block Diagram



(b) Signal Flow-Graph

$$t_s = 4/\zeta_s \omega_N \quad . \tag{17}$$

For a stable system then, the contours of lines of constant settling time of a second order system in the s-plane are

$$s = \zeta_s \omega_N \quad ,$$

or

$$s = -4/t_s \quad . \tag{18}$$

Accordingly, from one definition of the z-transformation, [2],

$$z = e^{Ts} \quad , \tag{19}$$

we can determine corresponding contours of constant settling time in the z-plane as,

$$z = e^{-4T/t_s} \quad , \tag{20}$$

which, since T and t_s are constants for any particular design, are circles centered at the origin of the z-plane of radius e^{-4T/t_s}.

Thus, the loci of constant ω_z contours on the parameter plane can be interpreted as loci of constant settling times; and, when ω_z and T are specified, the appropriate t_s can be uniquely determined. Solving equation (20) for t_s,

$$t_s = -4T/\ln \omega_z \quad . \tag{21}$$

Equation (21), however, must be interpreted in terms of the z-plane, discrete, characteristics. For example, for a step input, the minimum settling time obtainable is nT seconds, for an n-th order system, when all the closed loop roots are at the origin of the z-plane. Accordingly, when ω_z approaches zero in equation (21), t_s should approach nT. Thus, for proper interpretation, equation (21) must be written as

$$t_s = nT - 4T/\ln \omega_z \quad . \tag{22}$$

The particular conclusion of interest here is, in terms of the parameter plane, to minimize t_s one has only to minimize ω_z within the stable

region of the plane.

To demonstrate the effectiveness of the constant ω_z loci in aiding the designer meet settling time criteria, consider again the pure second order system of figure (1), whose characteristic equation is given on page 6. It is first necessary to define the system parameters, so let

$$\alpha \triangleq a_1 K T^2 \tag{23}$$

and

$$\beta \triangleq a_2 K T \quad .$$

By reference to equation (9) and the characteristic equation, then

$$\begin{array}{lll} d_0 = 2 & b_0 = 1 & c_0 = -2 \\ d_1 = -4 & b_1 = 1 & c_1 = 2 \\ d_2 = 2 & b_2 = 0 & c_2 = 0 \end{array}$$

which when substituted into equation (10) yields

$$\begin{array}{ll} B_1 = -U_{-1}(\zeta_z) & B_2 = \omega_z U_1(\zeta_z) \\ C_1 = 2U_{-1}(\zeta_z) & C_2 = 2\,\omega_z U_1(\zeta_z) \end{array}$$

$$D_1 = -2U_{-1}(\zeta_z) - 2\,\omega_z^2\, U_1(\zeta_z)$$

$$D_2 = -4\,\omega_z U_1(\zeta_z) + 2\,\omega_z^2\, U_2(\zeta_z) \quad .$$

These factors can now be substituted into equation (11) and the constant ω_z loci plotted with the aid of a desk calculator or a digital computer. Note that if a digital computer is used, as is most likely, a limiting process is required to remove the common factor of ω_z from the numerator and denominator of the parametric equations when attempting to locate the $\omega_z = 0$ point on the parameter plane. Figure (2) shows the resulting lines of constant ω_z on the parameter plane for this system.

From figure (2), minimum settling time is predicted when $\omega_z = 0$, or when

$$\alpha = 1.0 \quad ,$$

and

$$\beta = 1.5 \quad .$$

In order to determine the system response as predicted by equation (14), let

$$T = K = 1.0 \qquad ,$$

then $$a_1 = 1.0$$

and $$a_2 = 1.5 \text{ is required.}$$

Table 1 lists the calculated magnitude values of the states of the system at consecutive sampling intervals using these values of the two parameters. The time response of the states is plotted on figure (3). A unit step input is assumed.

TABLE 1

nT	$X_1(t)$	$X_2(t)$
0	0	0
1	0.5	1.0
2	1.0	0
3	1.0	0
4	1.0	0

Since in this example, n = 2, these results are in agreement with that predicted by equation (22), and the response is the minimum prototype, ripple free response.

In order to demonstrate the most important design advantages in using the parameter plane for feedback compensation; assume that, for some other reason, one of the feedback coefficients is constant, say a_1 is fixed at 2.25. Let K = T = 1.0, also be constant. With these constraints on the design problem, the parameter plane affords an excellent potential for setting the remaining degree of freedom to minimize the settling time, in so far as possible with feedback compensation alone. Referring to figure (2) for $\alpha = 2.25$, ω_z is minimum at $\omega_z = 0.5$ when $\beta = 1.875$. From equation (22),

Figure 2
Parameter Plane
For Pure Second
Order System

$\alpha = a_1 K T^2$

$\beta = a_2 K T$

Loci of Constant Magnitudes

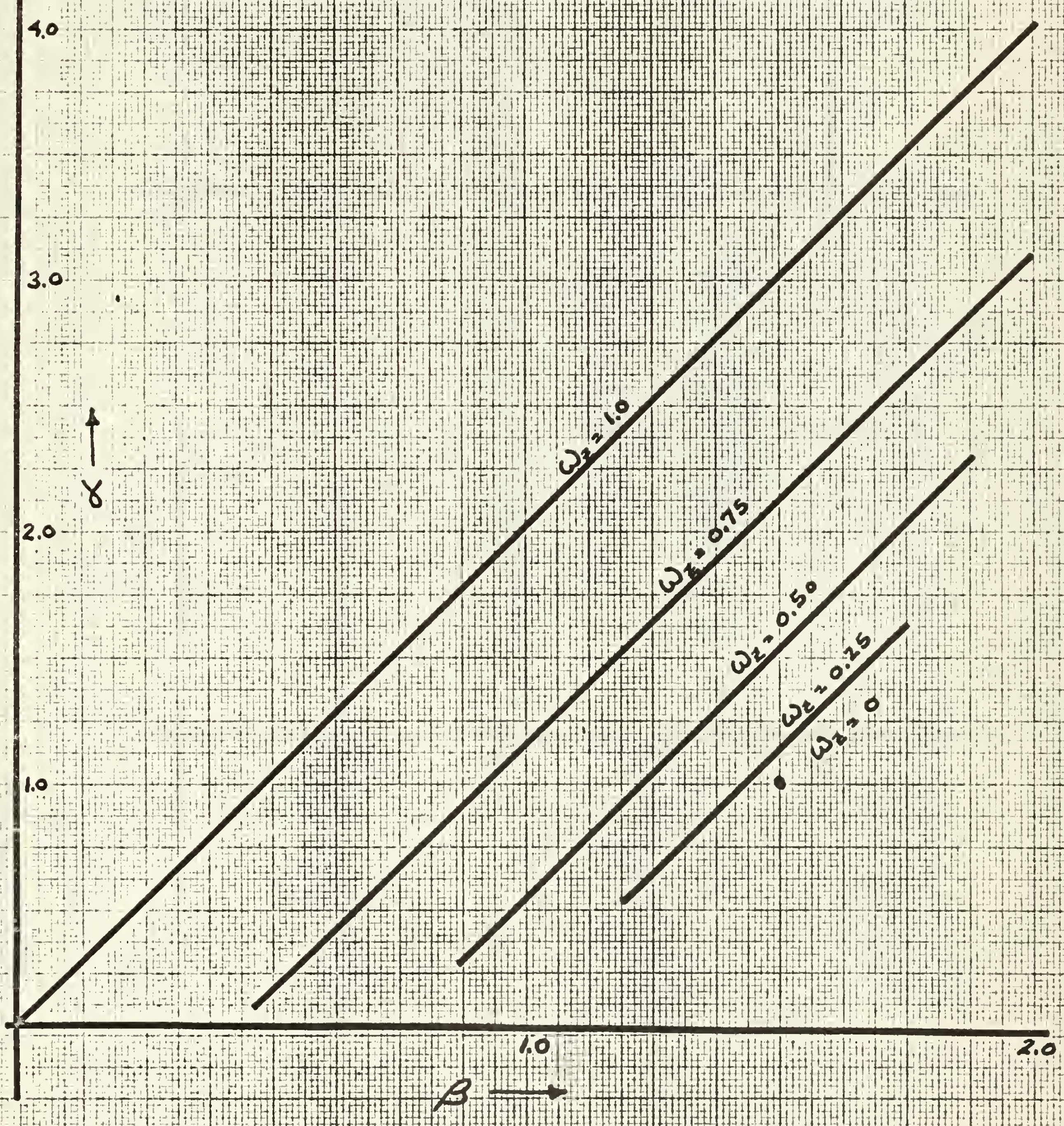
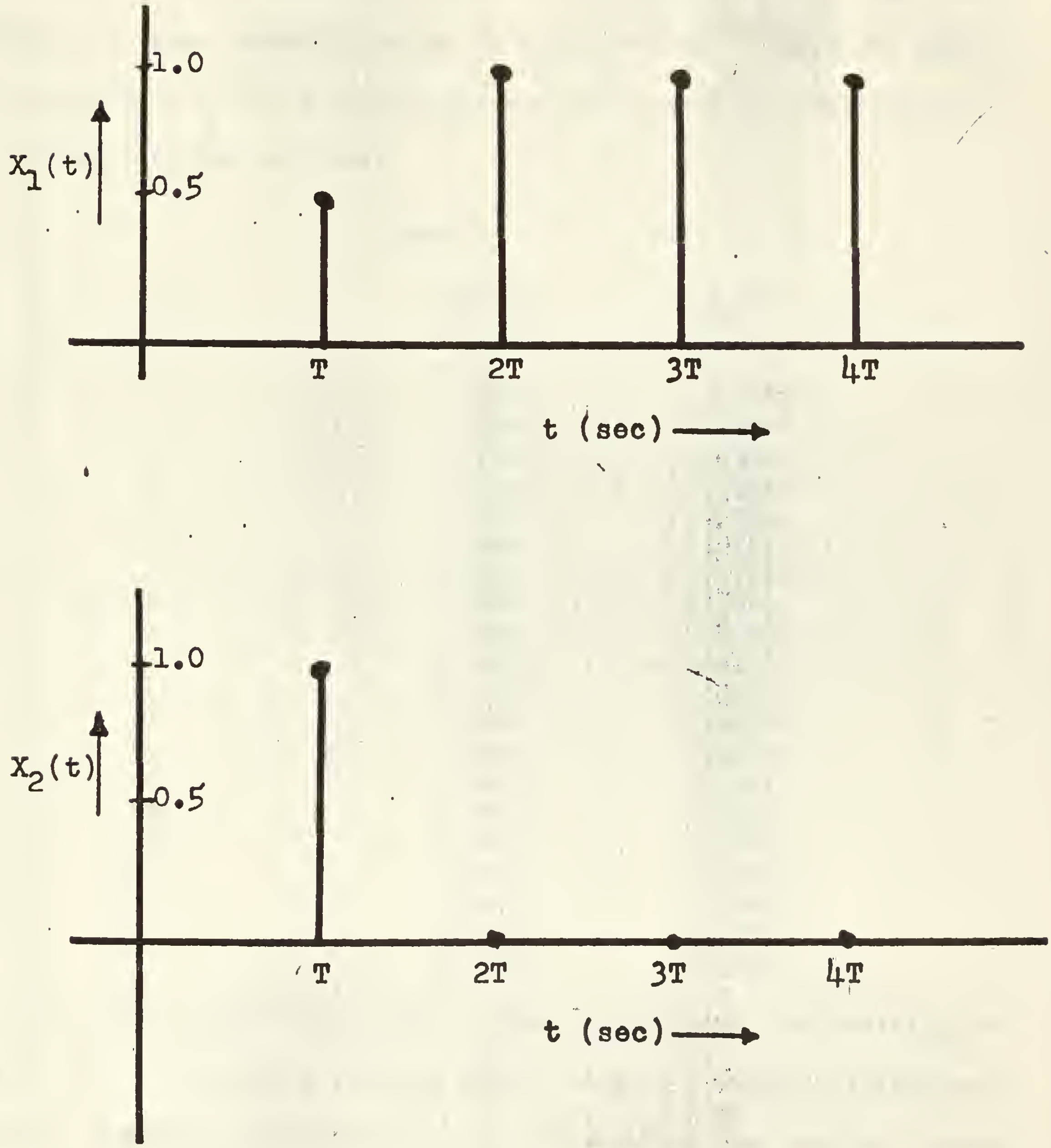


Figure 3
Response of Pure Second Order System
with Minimum Settling Time Compensation



$$t_s = 2T - 4T/\ln 0.5$$

or

$$t_s = 7.8T$$

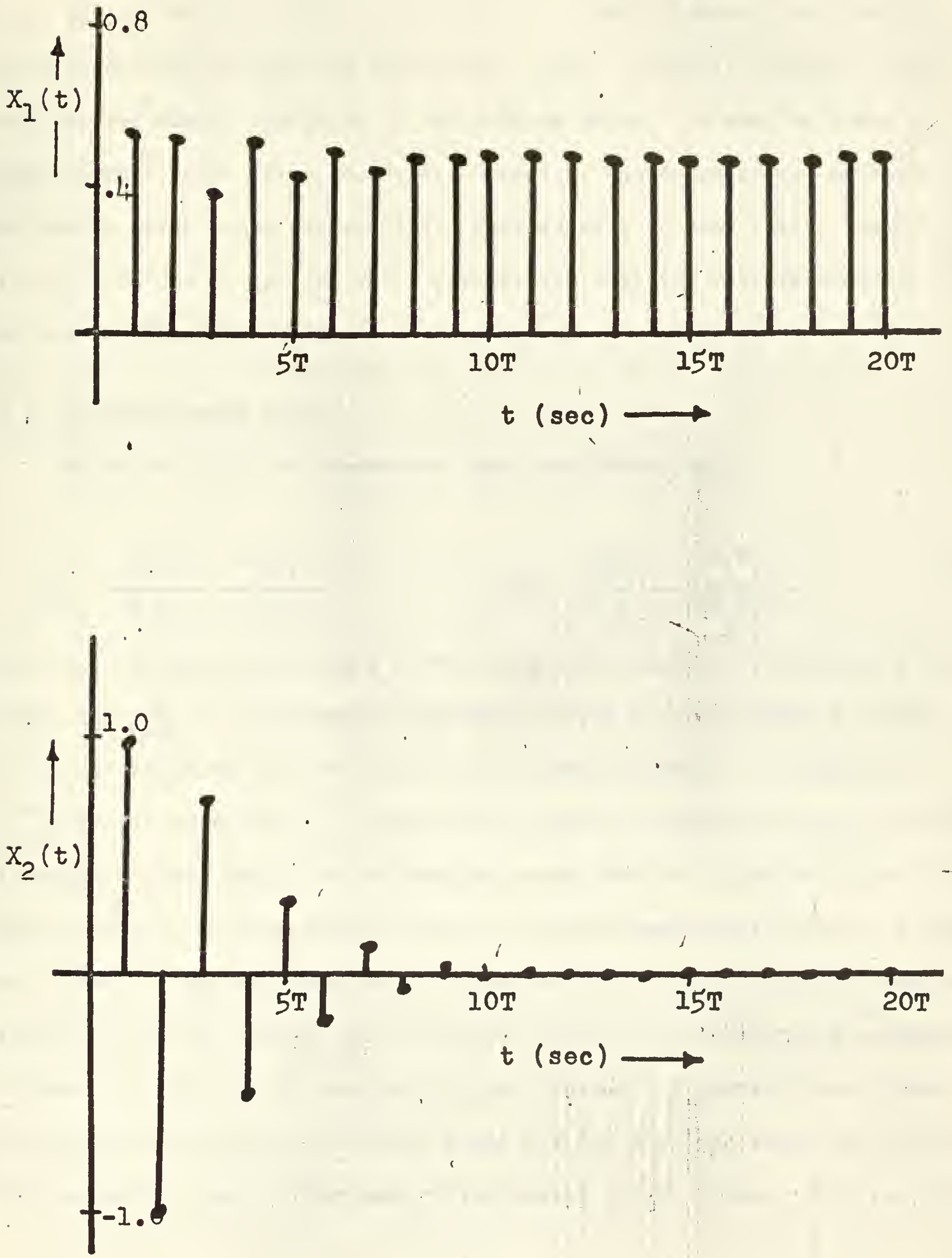
is the predicted settling time. Table 2 lists the state variable magnitudes at the indicated sampling interval in response to a unit step input utilizing the above parameter values in equation (14). Figure (4) shows the time response of the state variables of the system as a function of the sampling interval, or time.

TABLE 2

nT	$X_1(t)$	$X_2(t)$
0	0	0
1	0.5000	1.0000
2	0.5000	-1.0000
3	0.3750	0.7500
4	0.5000	-0.5000
5	0.4063	0.3125
6	0.4688	-0.1875
7	0.4297	0.1094
8	0.4531	-0.0625
9	0.4395	0.0352
10	0.4473	-0.0195
11	0.4429	0.0107
12	0.4453	-0.0059
13	0.4440	0.0032
14	0.4447	-0.0017
15	0.4443	0.0009
16	0.4445	-0.0005
17	0.4444	0.0003
18	0.4445	-0.0001
19	0.4444	0.0001
20	0.4444	-0.0000

The results indicated in Table 2 appear to discredit the validity of equation (22) in predicting settling time. Actually, equation (22) simply provides an engineering approximation to the settling time problem. Analysis of the results of Table 2 indicates the output position is within 1.95% of the final value after 8T seconds, although an actual zero velocity and final position of 0.4444 was not achieved until the 41st sampling interval.

Figure 4
Response of Pure Second Order System
with Constraints



A similar test was performed with $a_1 = 2.25$ and $a_2 = 1.50$ for $\omega_z = 0.8$. Equation (22) predicts the settling time as $t_s = 20T$, at which time the output had settled to within 3.3% of the final value. In that example, zero actual velocity did not occur until $t = 99T$. However, the settling time prediction of equation (22) remains within generally accepted good engineering design proximity of the desired value. It must be borne in mind however, that for a_1 different from 1.0, the conventional definition of system error cannot be applied. Fortunately, in many cases, input/output scaling can be applied which permits the position voltage feedback to be set at other than unity.

2.2 Constant Angle Loci.

Equation (11), the parametric equations describing

$$\alpha = \frac{C_1D_2 - C_2D_1}{B_1C_2 - B_2C_1} , \quad \beta = \frac{B_2D_1 - B_1D_2}{B_1C_2 - B_2C_1} \tag{11}$$

the loci of the complex roots on the parameter plane are functions of both ω_z and ζ_z. The constant magnitude curves were obtained by fixing ω_z and allowing ζ_z to vary as the mapping parameter. Similiarly, ζ_z can be fixed while ω_z varies from zero to infinity to map lines of constant z-plane angle for the complex roots onto the parameter plane. While the loci of this constant angle on the parameter plane does not have any direct usefulness comparable to the loci of constant magnitude and the determination of settling time, they do lead to a very important secondary property. With loci of constant ω_z and constant ζ_z drawn on the parameter plane, every point on the z-plane now has a unique point defined on the parameter plane in the case of the second order systems. The fact that

for higher order systems, several z-plane points may map onto one parameter plane point will be discussed later.

For the second order case, this one-to-one correspondence between points on the z-plane and points on the parameter plane is useful in conformal mapping of various loci from the z-plane to the parameter plane. Among the possibilities are the logarithmic constant damping spirals corresponding to lines of constant damping in the s-plane. Another use would be the mapping of lines of constant overshoot, [2] and [7], from the z-plane to the parameter plane. This z-plane mapping permits the designer to select feedback compensation which simultaneously satisfies overshoot and settling time criteria for the second order system.

2.3 Constant Real Root Loci.

In order to completely specify all possible z-plane root locations on the parameter plane, real root configurations must be allowed for as well as the complex root configurations. As indicated by equation (12), the loci of constant real roots are straight lines on the parameter plane. For the second order system, the closed loop poles are either both real or both complex. Therefore, the zone of real root description and the zone of complex root description cannot overlap, except at a critical point where the real root is about to emerge from the real axis.

This zone separation characteristic greatly simplifies the sketching of the parameter plane for the second order system. Fortunately, only one set of calculations is necessary. The constant ω_z curves must be calculated. However, since these loci of constant magnitude are straight lines, only a minimum amount of calculations is involved. In plotting the constant ζ_z curves, one has only to connect the coordinates of equal ζ_z on the ω_z curves. Then, to plot the lines of constant σ_z ,

since from equation (2) we can write,

$$\sigma_z = \omega_z \zeta_z \tag{24}$$

one has only to draw straight lines tangent to the $|\zeta_z| = 1.0$ line at the ω_z intersections. For example, a line drawn tangent to the $\zeta_z = -1.0$ line at the intersection of the $\omega_z = 0.5$ line describes the real root locations $\sigma_z = -0.5$. This sketching technique will not necessarily be extendable to the higher order systems unfortunately; however, it does increase the simplicity of the second order parameter plane.

Figure (5) is the parameter plane for the pure second order system of figure 2 which now has some of the loci of constant angle (solid lines) and some of the loci of constant real roots (broken lines) superimposed upon it. Note that this parameter plane configuration also accurately describes the system stability in terms of the two parameters.

2.4 Pole Effect.

In order to study the pole effect of the second order systems on the parameter plane, the system shown in figure (6) was studied. This system can also be represented by the vector-matrix equation (14), where in this case

$$\Phi = \begin{bmatrix} 1 - \frac{Ka_1}{P^2}(PT - 1 + \epsilon^{-PT}) & \frac{1}{P}(1 - \epsilon^{-PT}) - \frac{Ka_2}{P^2}(PT - 1 + \epsilon^{-PT}) \\ -\frac{Ka_1}{P}(1 - \epsilon^{-PT}) & \epsilon^{-PT} - \frac{Ka_2}{P}(1 - \epsilon^{-PT}) \end{bmatrix}$$

Figure 5
Parameter Plane for
Pure Second Order System -
(Complete)

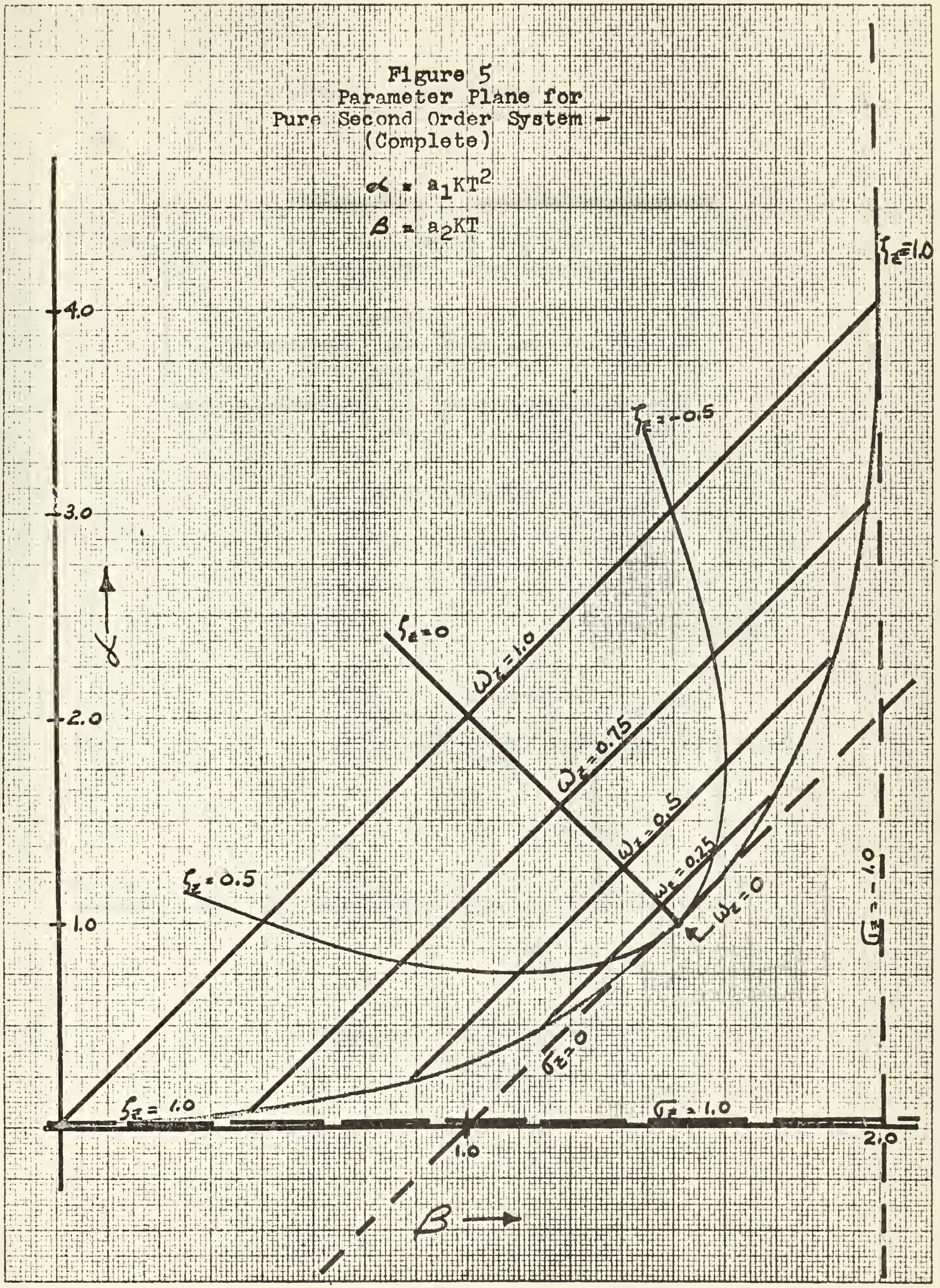
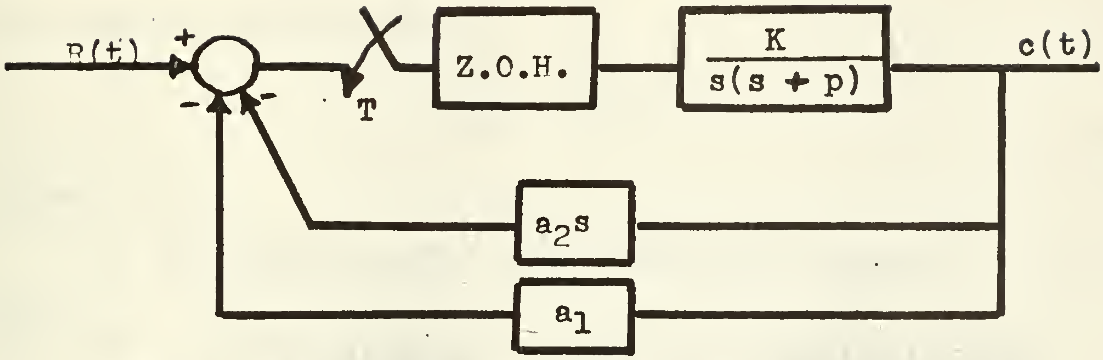
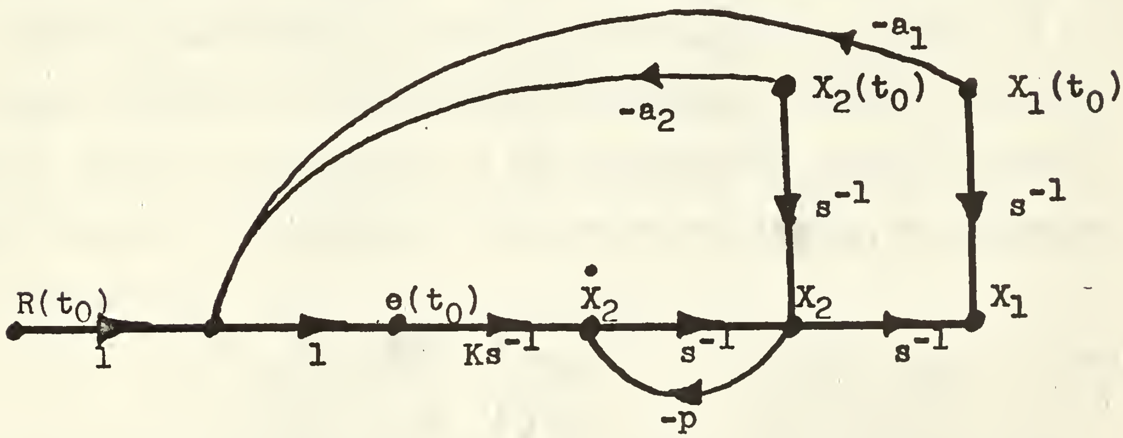


Figure 6
General Second Order
System Diagrams



(a) Block Diagram



(b) Signal Flow-Graph

and

$$\underline{\Delta} = \begin{bmatrix} \frac{K}{p^2}(pT - 1 + \epsilon^{-pT}) \\ \\ \frac{K}{p}(1 - \epsilon^{-pT}) \end{bmatrix}$$

The characteristic equation is

$$z^2 + A_1 z + A_0 = 0 \tag{25}$$

where

$$A_1 = Ka_1(T/p-1/p+e^{-pT}/p^2) + Ka_2(1/p-e^{-pT}/p)-1-e^{-pT}$$

and

$$A_0 = Ka_1(1/p^2-Te^{-pT}/p-e^{-pT}/p^2) + Ka_2(e^{-pT}/p-1/p)+e^{-pT}.$$

One important difference between the pure second order system and the system with damping becomes evident by inspecting the characteristic equation above. In the pure second order case, by judicious definition of the system parameters, equation (23), it was possible to produce a parameter plane as a function (indirectly) of all physical parameters, K, T, a_1, and a_2. By moving one of the poles into the left-half of the s-plane, it is no longer possible to obtain parameters involving T. However, the system gain, K, remains a common factor in the characteristic equation coefficients. Therefore, for the second order system with damping, we can define the parameters as

$$\begin{aligned} \alpha &\triangleq Ka_1 \\ \beta &\triangleq Ka_2 \end{aligned} \tag{26}$$

For further study of the pole effects, let p = T = 1.0, then from equation (9),

$d_0 = 0.368$	$b_0 = 0.264$	$c_0 = -0.632$
$d_1 = -1.368$	$b_1 = 0.368$	$c_1 = 0.632$
$d_2 = 1.0$	$b_2 = 0$	$c_2 = 0$

and from equations (10),

$$B_1 = -0.264U_{-1}(\zeta_z) \qquad B_2 = 0.368\,\omega_z U_1(\zeta_z)$$

$$C_1 = 0.632U_{-1}(\zeta_z) \qquad C_2 = 0.632\,\omega_z U_1(\zeta_z)$$

$$D_1 = -0.368U_{-1}(\zeta_z) - \omega_z^2 U_1(\zeta_z)$$

$$D_2 = -1.368\,\omega_z U_1(\zeta_z) + \omega_z^2 U_2(\zeta_z).$$

Figure (7) is the parameter plane for this system, with loci plotted for comparison with figure (5). Note that the shifting of one pole from the origin of the s-plane into the left-half plane has produced a counter-clockwise rotation and a compression of the upper semi-circle contours. This suggests that the further into the left half plane of the real pole (closer to the origin of the z-plane) the more sensitive the system response is to changes in a_2. Additionally, from figure (7), the extremities of the complex root region on the parameter plane have been extended, to the point where negative values of beta are now within the stable region. From figure (7), minimum prototype response to a step input should be expected for

$$\alpha = 1.58$$

$$\beta = 1.24$$

Consequently, there are two ways to achieve minimum prototype compensation. First, if K is fixed to some value other than 1.58, for example 1.0, then non-unity position feedback must be used. With K = 1.0, the magnitudes of the state variables at consecutive sampling intervals in response to a unit step input, calculated using equation (14), are shown by table 3. Note that this response is identical to that shown on figure (3), except for a constant scale factor, which is naturally equal to $1/a_1$.

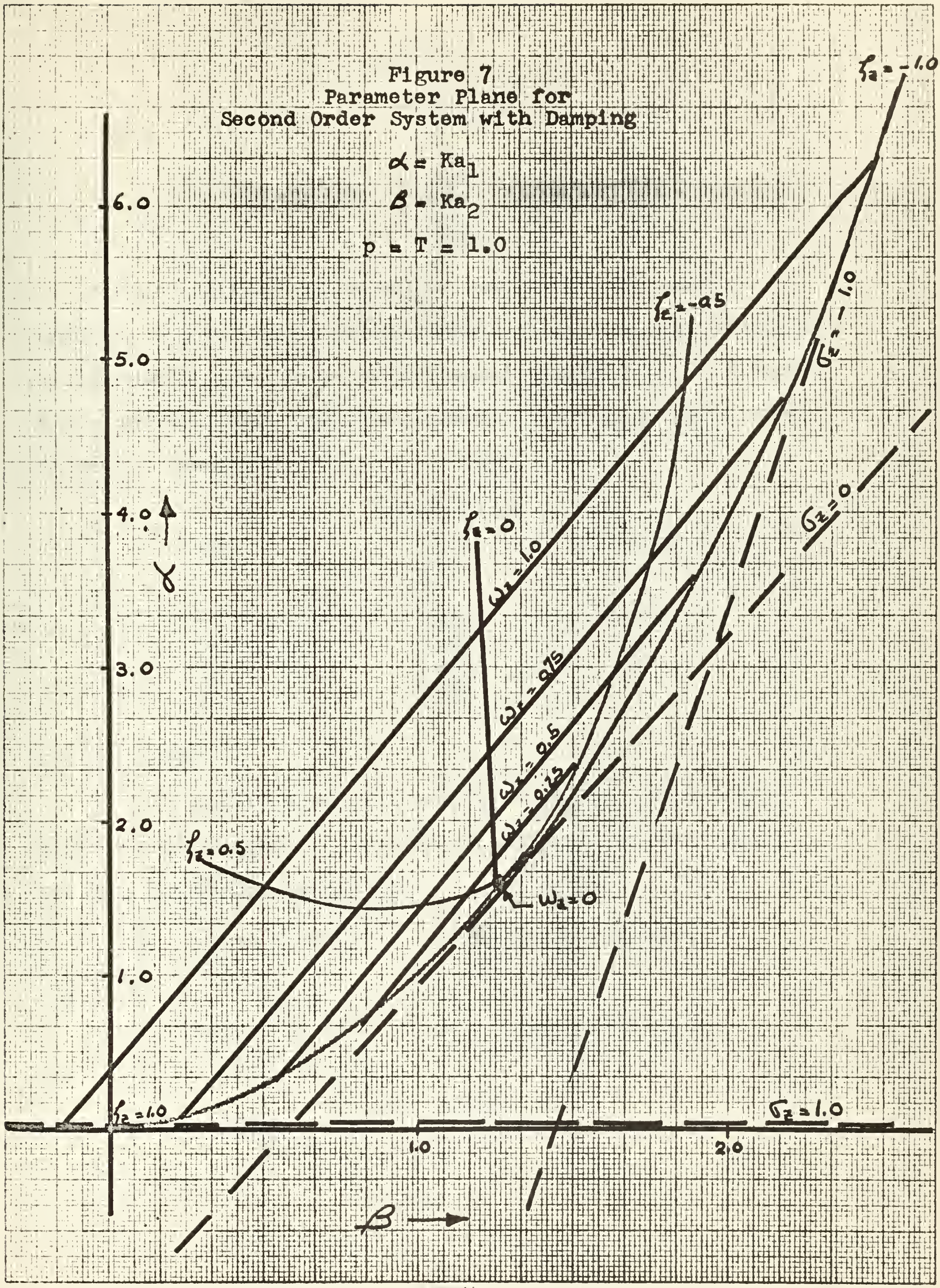


Figure 7
Parameter Plane for
Second Order System with Damping
$\alpha = Ka_1$
$\beta = Ka_2$
p = T = 1.0
6.0
5.0
4.0
3.0
2.0
1.0
1.0
2.0
γ
β
$\zeta_z = -1.0$
$\zeta_z = -0.5$
$\zeta_z = 0$
$\zeta_z = 0.5$
$\zeta_z = 1.0$
$\sigma_z = -1.0$
$\sigma_z = 0$
$\sigma_z = 1.0$
$\omega_z = 1.0$
$\omega_z = 0.75$
$\omega_z = 0.5$
$\omega_z = 0.25$
$\omega_z = 0$

TABLE 3

nT	$X_1(t)$	$X_2(t)$
0	0	0
1	0.368	0.632
2	0.632	0
3	0.632	0
4	0.632	0

For the particular case where the conventional error definition is required, to obtain the minimum prototype response, K must be restricted to the value of 1.58. Hence, the corresponding value for a_2 is 1.24/K or 0.785. The calculated response using these parameter settings is shown by table 4, assuming a unit step input again.

TABLE 4

nT	$X_1(t)$	$X_2(t)$
0	0	0
1	0.582	1.0
2	1.000	0
3	1.000	0
4	1.000	0

2.5 Zero Effect.

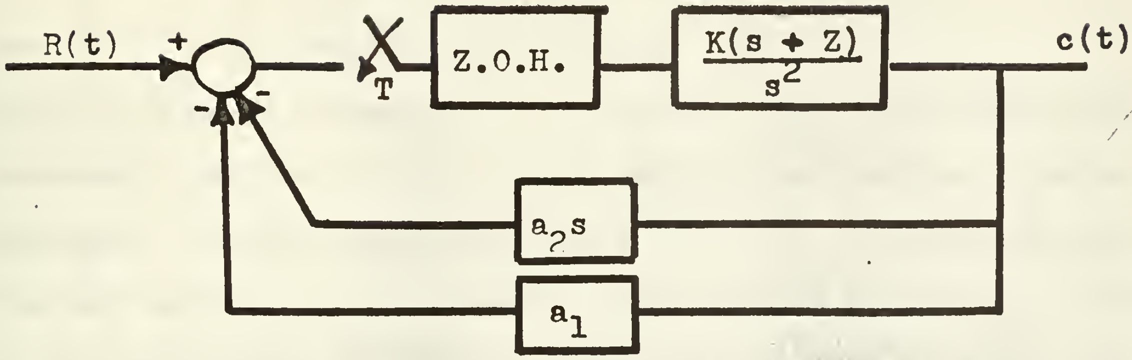
To study the effect of a real zero on the second order parameter plane, the system shown in figure (8) was analyzed. For this system, the state transition matrices are:

$$\underline{\Phi} = \begin{bmatrix} 1 - Ka_1(T + ZT^2/2) & T - Ka_2(T + ZT^2/2) \\ \\ - Ka_1(1 + ZT) & 1 - Ka_2(1 + ZT) \end{bmatrix}$$

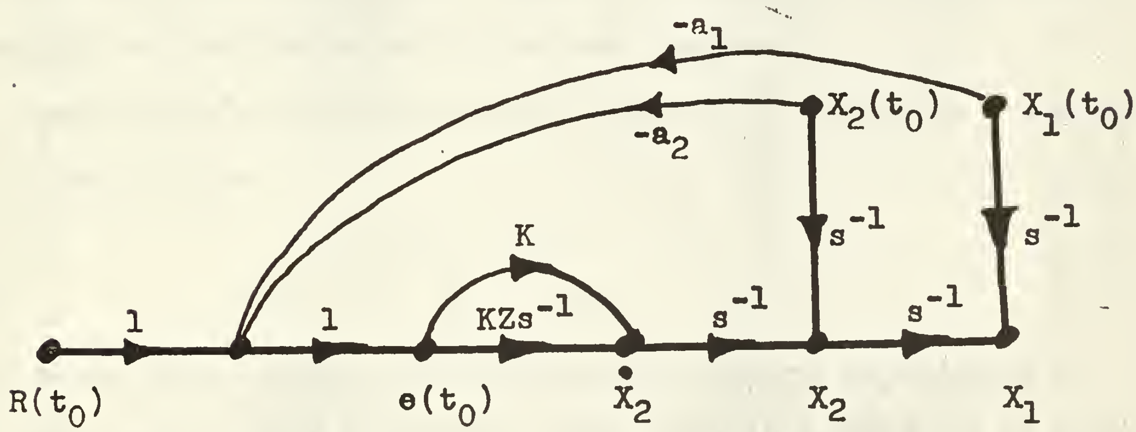
and

$$\underline{\Delta} = \begin{bmatrix} K(T + ZT^2/2) \\ K(1 + ZT) \end{bmatrix}$$

Figure 8
Second Order System
with Zero,
Diagrams



(a) Block Diagram



(b) Signal Flow-Graph

Again, the characteristic equation is

$$z^2 + A_1 z + A_0 = 0 \qquad ,$$

where

$$A_0 = Ka_1(ZT^2 - 2T) - Ka_2(2ZT + 2) + 1$$

and
$$A_1 = Ka_1(ZT^2 + 2T) + Ka_2(2ZT + 2) - 2 \quad .$$

Again, a choice of parameters for the parameter plane can be made as was performed in section 2.4. For further study, Z and T must be fixed in magnitude. For this example then, let $Z = T = 1.0$. Following the now familiar calculations, the parameter plane of figure (9) results. Similiar to the results shown for the pole effect, the parameter plane loci have been rotated counter-clockwise from the pure second order configuration of figure (5). Apparently this counter-clockwise rotation can be associated with the relative stability of the system, as both the system with the zero and the system with the left-half s-plane pole are inherently more stable than the pure second order servo. However, no measure for quantitatively describing this relative stability has been discovered.

From figure (9), minimum prototype, ripple free response to a step input is predicted for

$$\alpha = 0.5$$

$$\beta = 0.125$$

Again there are two choices for the physical parameters in achieving this compensation. If K is variable, unity position feedback may be used. If K is fixed, then non-unity position feedback is required. Responses similiar to that achieved in the last section result for each of these options.

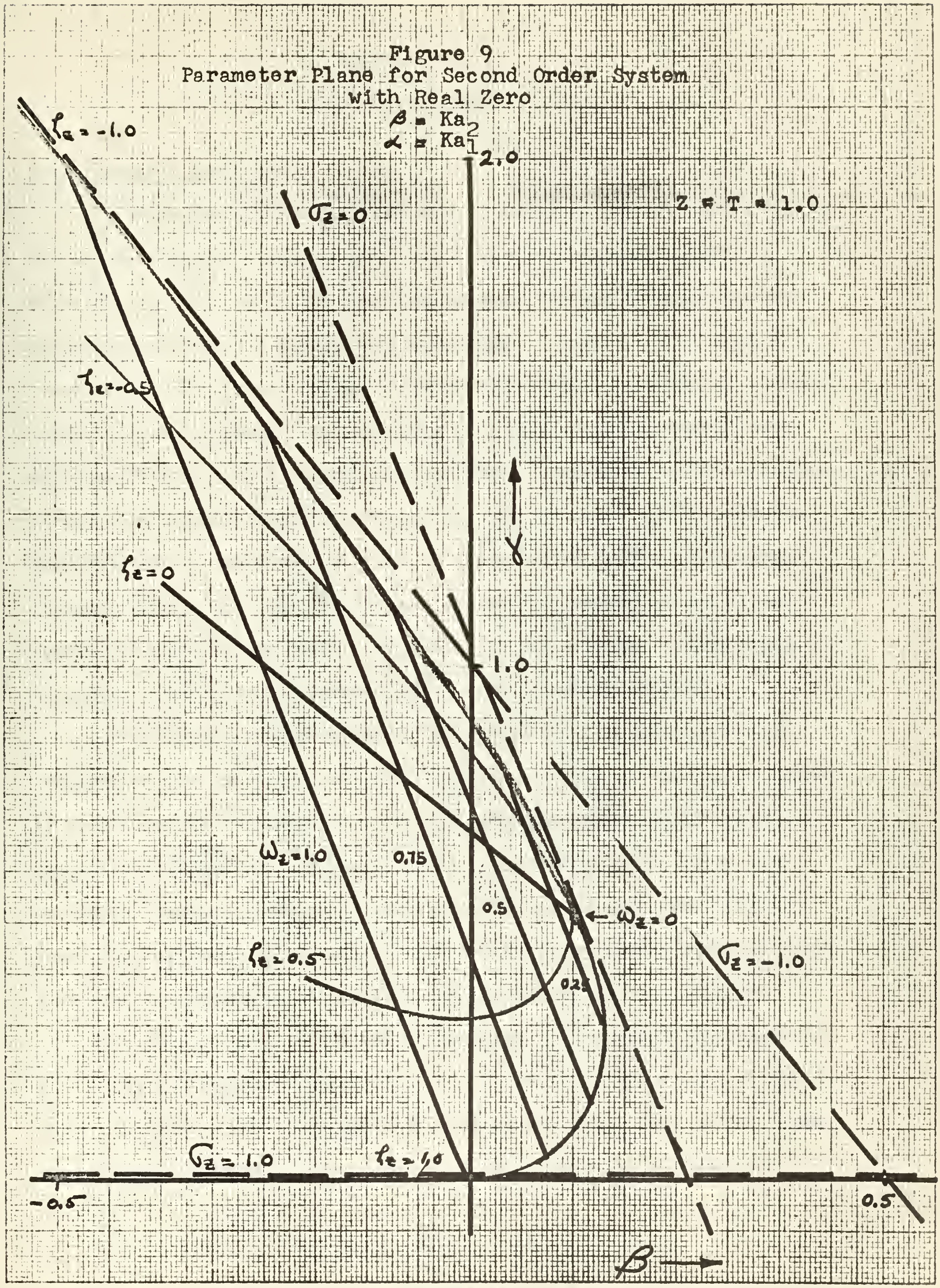


Figure 9
Parameter Plane for Second Order System
With Real Zero

CHAPTER III

HIGHER ORDER SYSTEMS

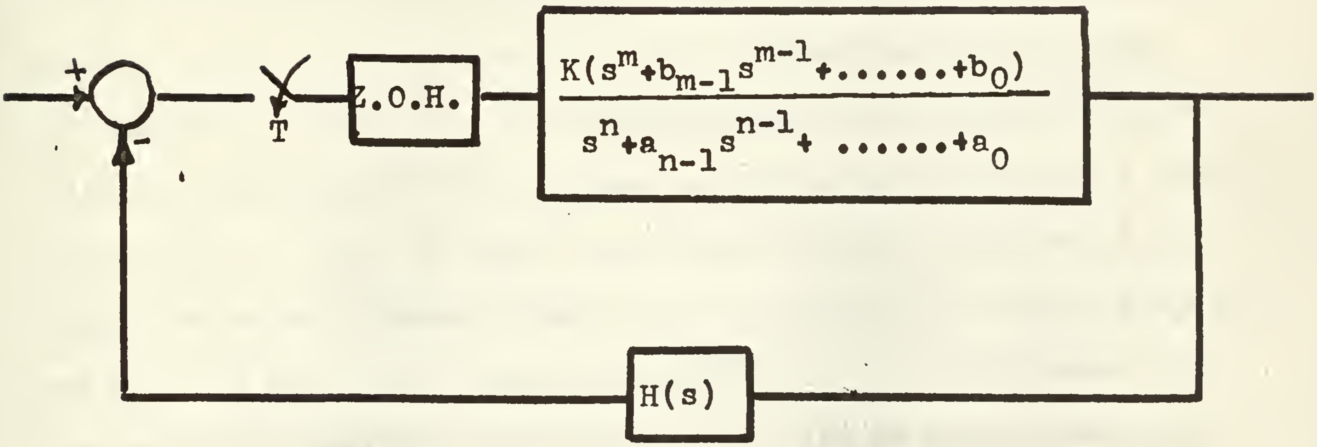
3.1 Parameter Limitations.

It can be readily demonstrated that in order to achieve complete freedom in locating the closed loop roots of an n-th order characteristic equation, the control system error signal must contain feedback intelligence on each of the n dynamically independent "states" of the system. Consequently, the system analog or mathematical model consists of n parameters and the parameter space becomes an n-space. Unfortunately, the existing Mitrovic, or algebraic, methods have their basis in the solution of two simultaneous equations, equations (5a) and (5b). This restricts the mathematical treatment of any n-th order system to only two degrees of freedom. While this reduction in the degrees of freedom restricts the ability to position the closed loop poles in the z-plane, a large class of control systems can be reduced to the two parameter problem.

Typically, the gain setting, K, in figure (10) is dictated by the requirements for steady state error. A second degree of freedom is normally eliminated by the adoption of the conventional error definition which demands unity feedback. Finally, in most physical systems, noise free derivative signals higher than the second derivative are unobtainable. Thus, the only degrees of freedom for feedback compensation typically available are the magnitudes of the first and second derivative feedback.

With this restriction on the positioning of the closed loop roots in mind then, the feedback compensation of systems with order greater than two can be designed on the parameter plane using techniques already discussed. Two additional features of the parameter plane must now be considered.

Figure 10
Typical Block Diagram for
n-th Order Sampled-Data System



$m \leq n$

Unfortunately, at the present time, these two features tend to reduce the effectiveness of the Mitrovic Method. However, it is believed that this reduced effectiveness is still an improvement over the existing one parameter design techniques.

3.2 Maximum Modulus Uncertainity.

The parametric equations describing the parameter plane loci were derived from the system characteristic equation. Any specified location (M-point) on the resulting parameter plane can be interpreted as the values of the two parameters which will put a closed loop root at the specified position of the z-plane. Therefore, for a typical fourth order system for example, the closed loop root configurations shown by figures(11a) and (11b) each correspond to a unique point on the parameter plane, each lying on the $\omega_z = 1.0$ loci. In order to distinguish between the different configurations on the parameter plane, it is also necessary to plot the $\omega_z = 0.5$ and $\omega_z = 1.5$ loci. The intersection of two of the loci of constant ω_z then adequately describes the location of each pair of complex roots. If figure (12) represents the parameter plane for the fourth order system of figure (11), then the root location depicted by figure (11a) corresponds to point A, while the configuration of figure (11b) corresponds to point B on the parameter plane.

Thus, for an n-th order system, the modulus of each real root or each pair of complex roots is completely determined on the parameter plane. The accuracy of that determination, however, is directly related to the number of loci of constant ω_z drawn. In the second order case, it was fairly easy to extrapolate between adjacent loci. As the order of the characteristic equation increases, the difficulty of making this extrapolation increases. It should be noted that this problem of interpreting

Figure 11
Typical z-Plane Closed Loop Root
Configuration for
Fourth Order System

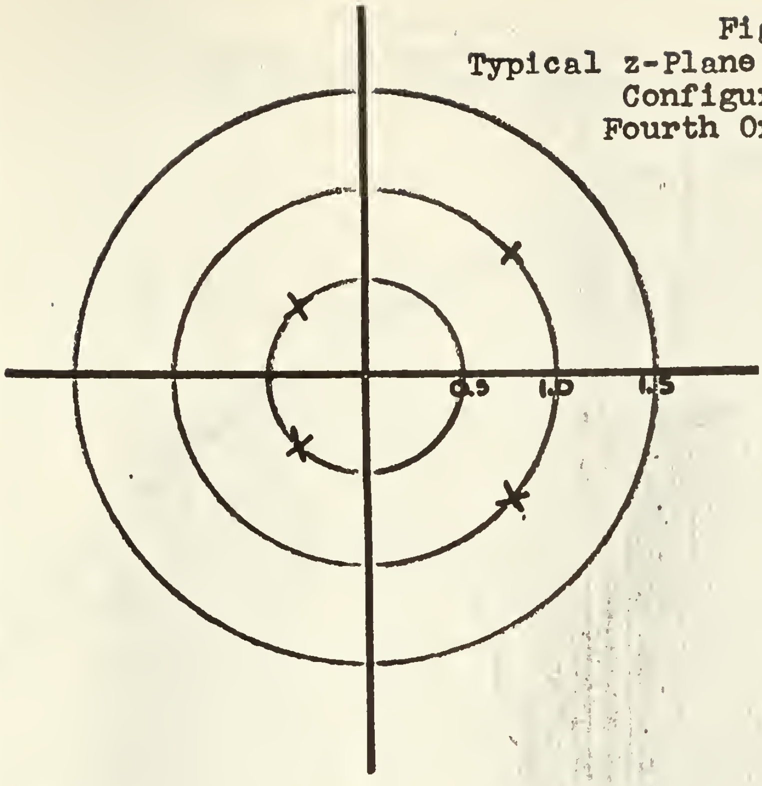


Figure (11a)

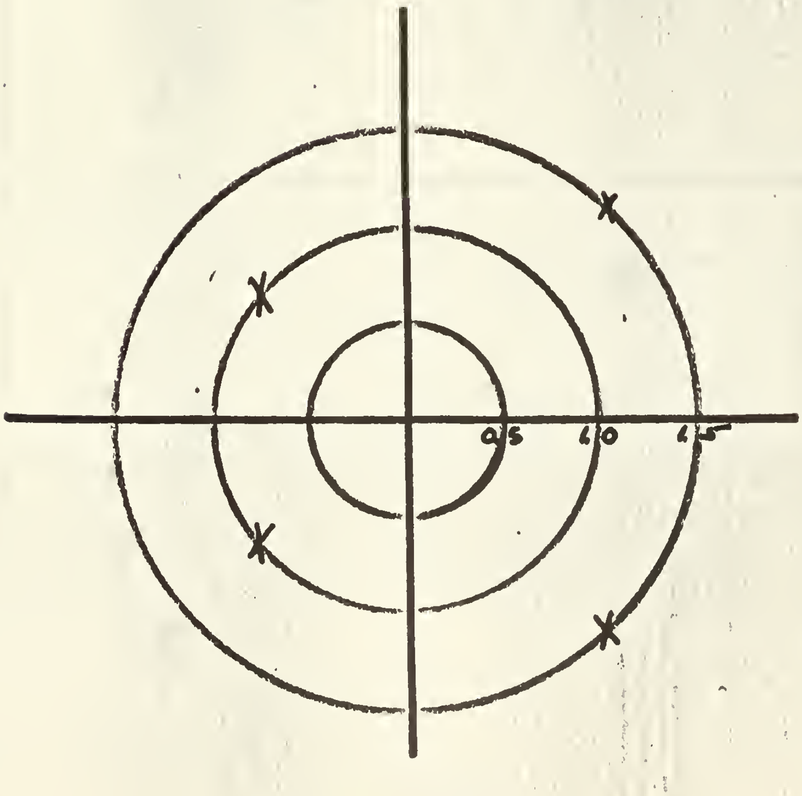
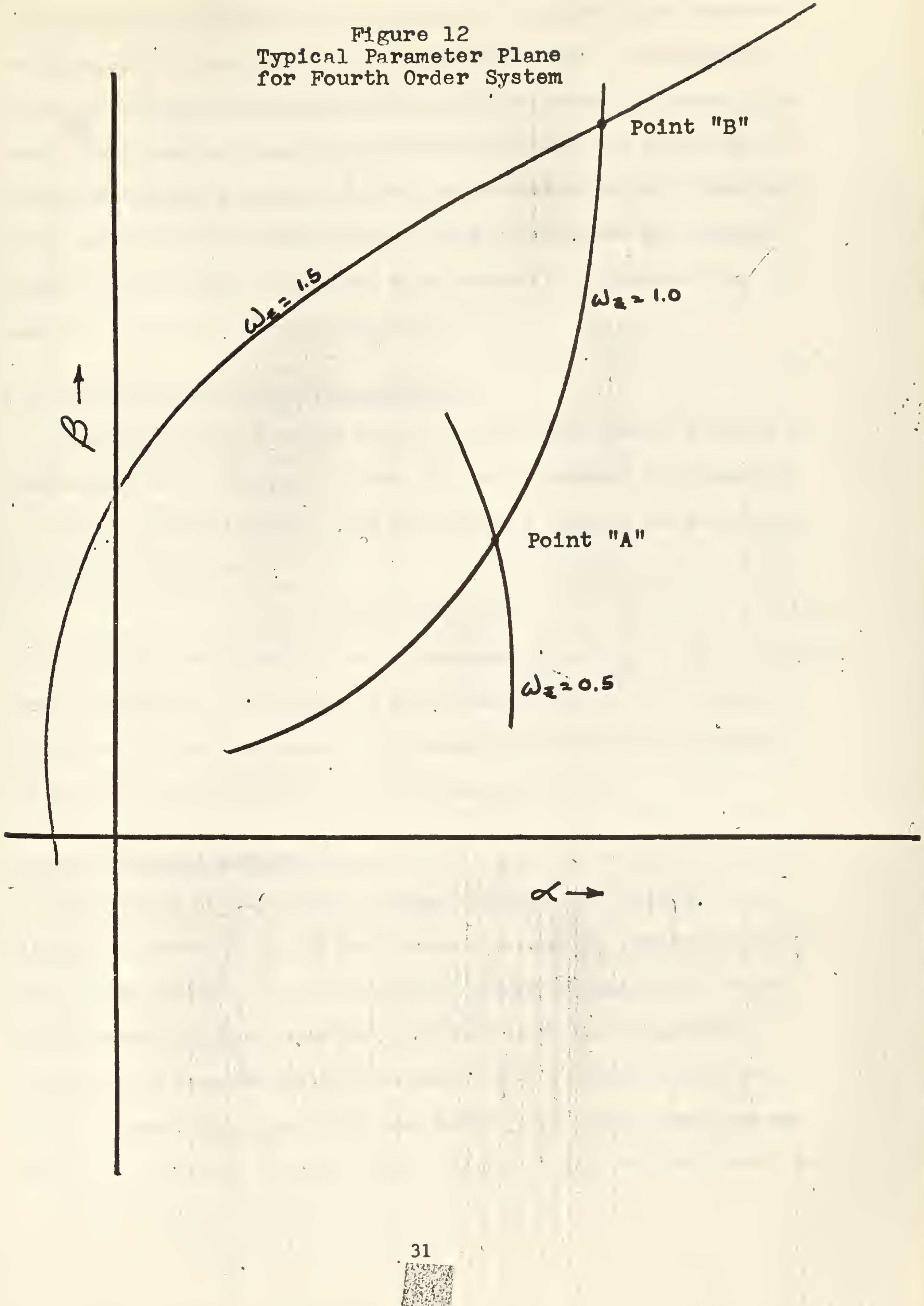


Figure (11b)

Figure 12
Typical Parameter Plane
for Fourth Order System



intersections of constant loci also exists in the second order case when the plant gain is insufficient to produce complex roots. Uncertainity arises as the order of the system increases. For example, given a fifth order system, and a corresponding parameter plane M-point defined by the intersection of two constant ω_z loci and one real root loci. Obviously, this M-point must determine two pair of complex roots and one real root. However, with existing techniques, it is impossible to associate any specific ω_z with a specific complex root.

3.3 Discontinuities on the Parameter Plane.

A second feature which may exist in higher order systems involves discontinuities on the constant ω_z and ζ_z loci. Equation (11) describes the parametric equations for alpha and beta as a ratio of two polynomials. Consequently, whenever

$$B_1C_2 - B_2C_1 = 0, \tag{26}$$

a discontinuity exists on the loci of constant ω_z or ζ_z. At the present time, the physical significance of this discontinuity is not understood, in fact one may not even exist. Its presence, however, does complicate the sketching and interpretation of the parameter plane.

3.4 Dominance and Settling Time.

In the case of the linear, continuous system, the response of a higher order system can often be discussed in terms of a dominant pair of roots in the s-plane. This is a pair of complex conjugate roots whose residue and/or transient settling time leads to by far the greatest portion of the observed physical response. In the s-plane, these roots have an argument and a real part considerably less in magnitude than any other pair of complex, or real, roots. If such a pair of roots exist, the

response can be quite accurately predicted in terms of a second order approximation and the settling time predicted by equation (17).

Similiarly, for the sampled-data system, the dominance of the system response can be discussed in terms of the pair of complex roots, or real root, whose modulus or argument is considerably greater than that of any other root, real or complex. If such a root exists, the settling time can be closely approximated by a slight modification to equation (22) such that

$$t_s = nT - 4T/\ln(\quad)_{max} \quad . \tag{27}$$

When the dominance cannot be clearly determined by the difference in the various arguments, equation (27) becomes less and less accurate in predicting the system settline time.

Alternatively, the minimization of the settling time can be achieved qualitatively by minimizing the modulus of each root or pair of complex roots. That is to say, compensation must be chosen which will place all the z-plane roots within a circle of minimum radius. If this circle of minimum radius happens to be the origin of the z-plane, then minimum prototype response is guaranteed. If, however, all the z-plane roots cannot be placed at the origin due to the limitations on the degrees of freedom available, minimum settling time will be somewhat greater than minimum prototype, and will not be ripple free. On the parameter plane, this minimization of the settling time involves the examination of the intersections of the constant ω_z loci for each permissible M-point, and selecting that one which does place all the roots within the circle of minimum possible radius. Observations of the resulting arguments will enable the designer to determine if equation (27) is then valid or not.

3.5 Third Order Example.

Consider the system shown on figure (13). This system can be represented by the signal flow graph of figure (14), and the vector-matrix equation

$$\underline{X}\left[(k+1)T\right] = \Phi\,\underline{X}(kT) + \underline{\Delta}\,e(kT) \quad ,$$

where

$$\Phi = \begin{bmatrix} 1 & (3/2 - 2e^{-T} + 1/2e^{-2T}) & (1/2 - e^{-T} + 1/2e^{-2T}) \\ 0 & (2e^{-T} - e^{-2T}) & (e^{-T} - e^{-2T}) \\ 0 & (-2e^{-T} + 2e^{-2T}) & (-e^{-T} + 2e^{-2T}) \end{bmatrix}$$

$$\underline{\Delta} = \begin{bmatrix} T/2 - 3/4 + e^{-T} - 1/4(e^{-2T}) \\ 1/2 - e^{-T} + 1/2(e^{-2T}) \\ e^{-T} - e^{-2T} \end{bmatrix}$$

and, $e(kT) = R(kT) - X_1(kT) - a_1X_2(kT) - a_2X_3(kT)$.

The characteristic equation, evaluated at T = 1 second is

$$Z^3 + A_2Z^2 + A_1Z + A_0 = 0 \quad ,$$

with $A_2 = 0.1998a_1 + 0.2325a_2 - 1.4191$

$A_1 = -0.1263a_1 - 0.4650a_2 + 0.7238$

and $A_0 = -0.0734a_1 + 0.2325a_2 - 0.0310$.

Observe that it is no longer possible, as was done for the second order systems, to include the plant gain, K, in the definition of the system parameters. This combination of the gain, K, with the feedback coefficients is only possible so long as each coefficient in the numerator of the product G(s)H(s) is some product containing a variable feedback gain. In the second order case,

$$G(s)H(s) = \frac{K(a_1 + a_2s)}{D(s)} \quad .$$

The characteristic equation can be formed as

$$1 + \text{z-transform of } \left[G(s)H(s)\right] \quad .$$

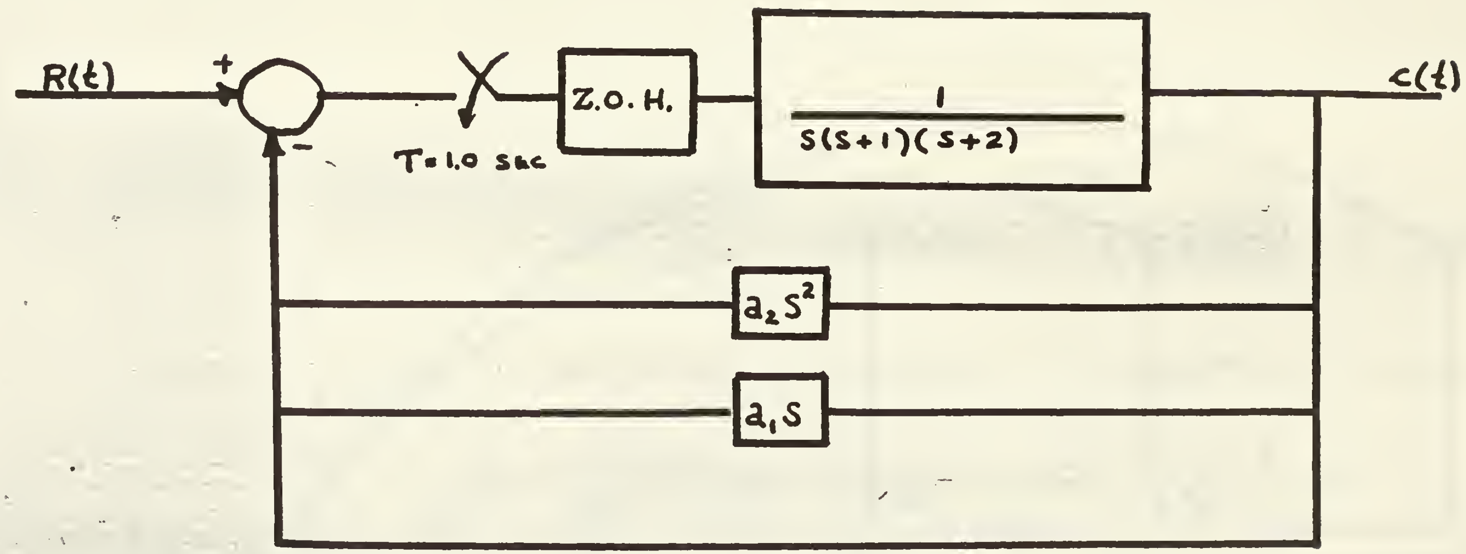


Figure 13
Block Diagram of
Third Order System

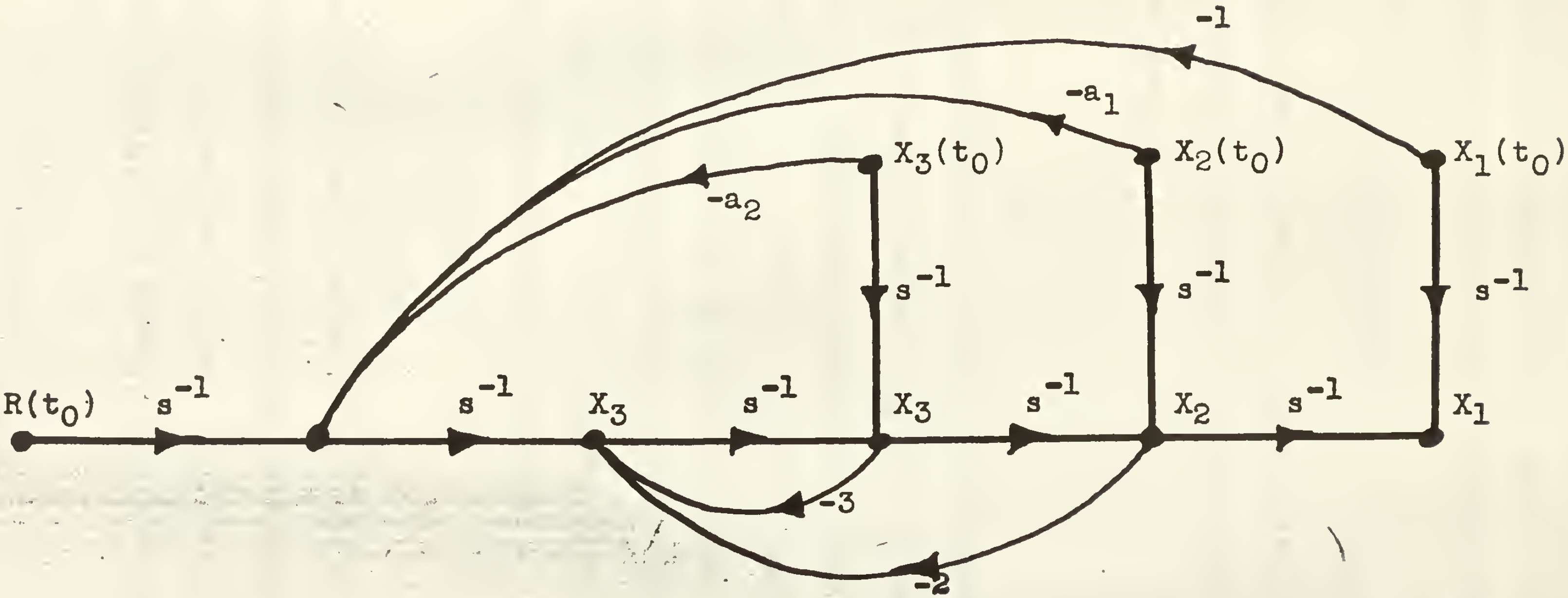


Figure 14
Signal Flow-Graph
for
Third Order Example

Thus, the only place that the feedback coefficients occur in the characteristic equation, they are each multiplied by the gain, K. However, for this third order example, the produce G(s)H(s) has the form,

$$G(s)H(s) = \frac{K(1 + a_1 s + a_2 s^2)}{D(s)} \quad .$$

In the characteristic equation then, K is a multiplier of terms other than those involving the feedback coefficients, a_1 and a_2. For the example, we are restricted to defining

$$\alpha \triangleq a_1$$
$$\beta \triangleq a_2$$

Following the usual manipulations, the third order parameter plane, figure (15), was achieved. No lines of constant ζ_z were included on the parameter plane, only to avoid unnecessary cluttering. Observe that the separation of complex root regions and real root regions no longer exists; which should be expected, as the root locus always has one real root. Also note that the region of stability is that region of the parameter plane which is enclosed by the contours $\omega_z = 1.0$, $\sigma_z = -1.0$, $\sigma_z = 1.0$. From figure (15), minimum settling time using feedback compensation is predicted when

$$a_1 = 1.61$$
$$a_2 = 0.54$$

The circle of minimum radius enclosing the roots is $\omega_z = 0.3$.

Table 5 lists the calculated magnitudes of the state variables at consecutive sampling intervals in response to a unit step input using these values for the feedback coefficients. From Equation (27) then, minimum settling time is calculated as

$$t_s = 3T - 4T/\ln 0.3$$
$$= 6.3T \quad .$$

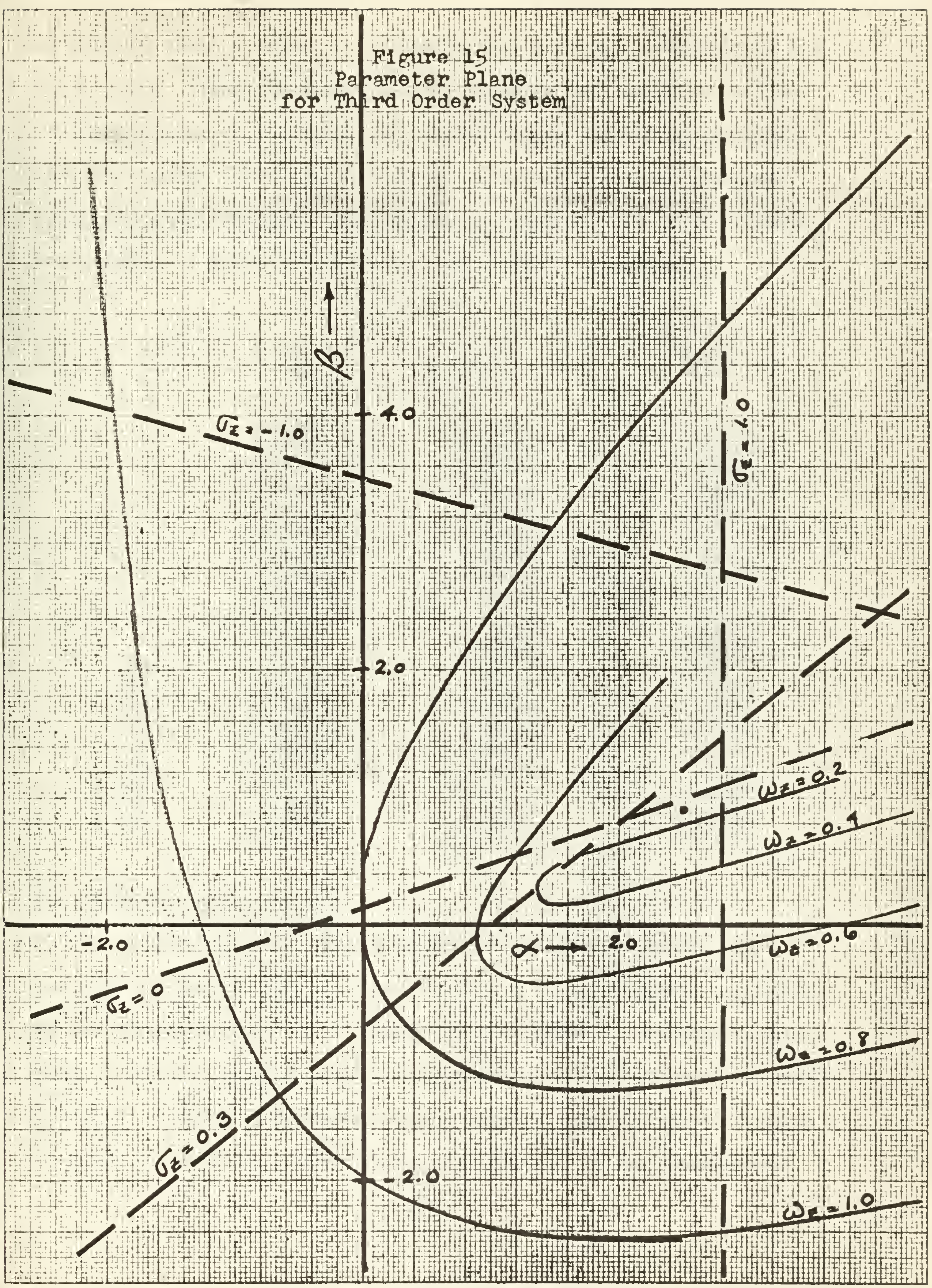


Figure 15
Parameter Plane
for Third Order System
β
α
4.0
2.0
-2.0
2.0
-2.0
σz = -1.0
σz = 1.0
σz = 0
σz = 0.3
ωz = 0.2
ωz = 0.4
ωz = 0.6
ωz = 0.8
ωz = 1.0

At time 7T, table 5 shows that the output has settled to within 5% of the final value. Figure 16 shows the state variable time response for this minimum settling time compensation. These results verify the effectiveness of equation (27) in predicting the servo settling time to a good engineering approximation, even with the dominance of the roots at maximum modulus, as discussed in section 3.4, somewhat uncertain.

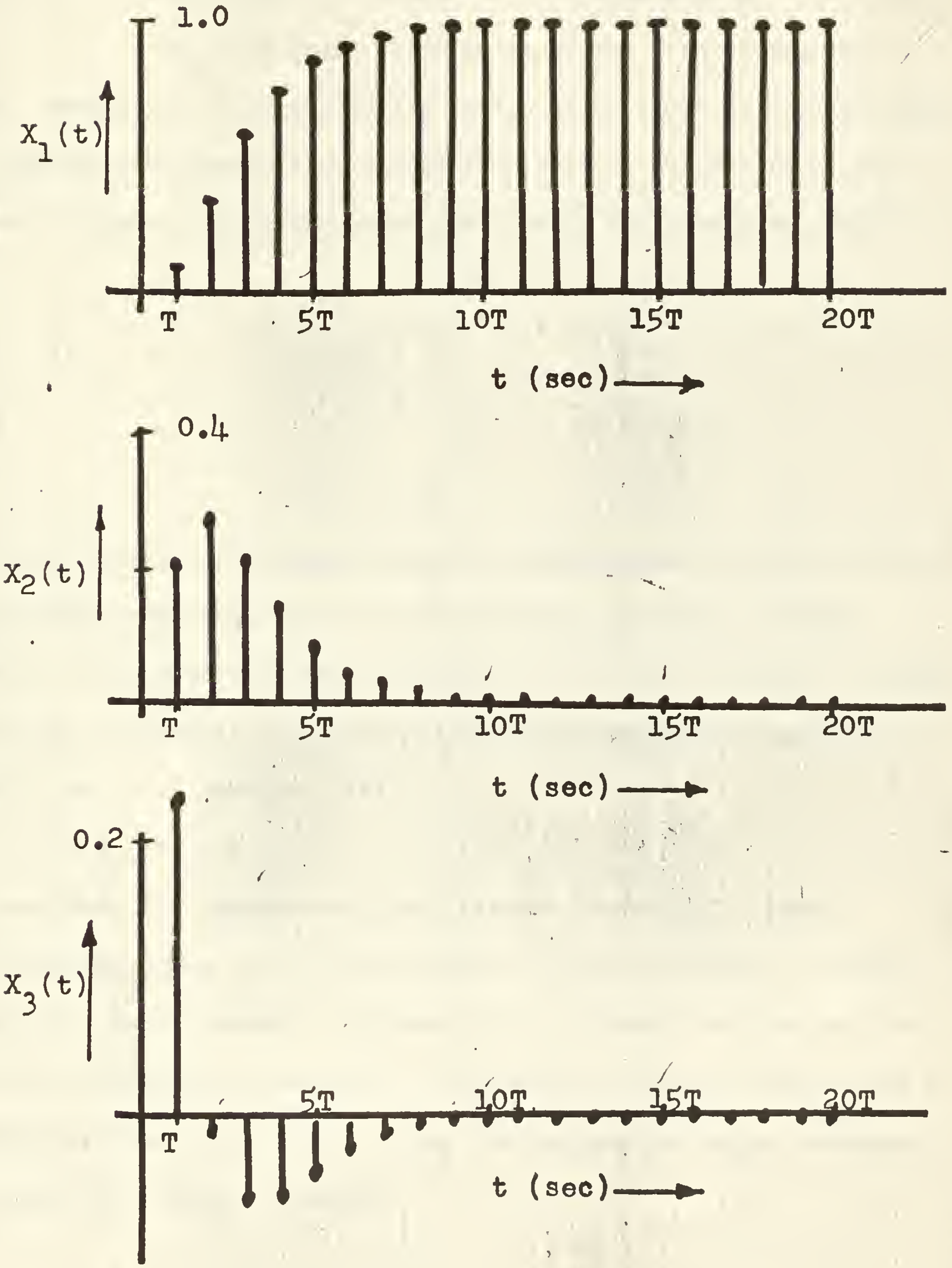
TABLE 5

nT	$X_1(t)$	$X_2(t)$	$X_3(t)$
0	0	0	0
1	0.08405	0.19979	0.23254
2	0.33610	0.26768	-0.00653
3	0.57736	0.20644	-0.06887
4	0.74605	0.13340	-0.05968
5	0.85111	0.08048	-0.03964
6	0.91357	0.04724	-0.02411
7	0.95002	0.02743	-0.01419
8	0.97114	0.01586	-0.00824
9	0.98334	0.00916	-0.00477
10	0.99039	0.00529	-0.00275
11	0.99445	0.00305	-0.00159
12	0.99680	0.00176	-0.00092
13	0.99815	0.00102	-0.00053
14	0.99893	0.00059	-0.00031
15	0.99938	0.00034	-0.00018
16	0.99965	0.00020	-0.00010
17	0.99980	0.00011	-0.00006
18	0.99988	0.00007	-0.00003
19	0.99993	0.00004	-0.00002
20	0.99999	0.00001	-0.00001

3.6 Algebraic Design Methods.

One simple and quite obvious method for designing feedback compensation for minimum prototype response should not be overlooked. In determining the system characteristic equation as a function of the feedback variables, which must be done to apply the Mitrovic Method, an alternate method arises for determining minimum prototype compensation. To obtain minimum prototype response for a given system, the function of any compensation

Figure 16
State Variable Time Response for Parameter Plane Compensated Third Order System
$a_1 = 1.61$
$a_2 = 0.54$



must be to drive all the closed loop roots to the origin of the z-plane. The characteristic equation of an n-th order system, such as shown on figure (10), is

$$z^n + A_{n-1} z^{n-1} + \ldots\ldots\ldots + A_0 = 0 \tag{28}$$

where

$$A_i = B_{0i} a_0 + B_{1i} a_1 + \ldots\ldots + B_{ni} a_n + B_n$$

and $\qquad a_j$ = feedback coefficient of the i-th derivative feedback. Therefore, to algebraically design the minimum prototype compensation, thereby positioning each closed loop root at the origin of the z-plane, one has only to simultaneously solve the n non-homogeneous equations

$$\begin{aligned} A_{n-1} &= 0 \\ A_{n-2} &= 0 \\ &\vdots \\ A_0 &= 0 \end{aligned} \quad .$$

The solution to these simultaneous equations determines the values of the n feedback parameters required to produce minimum prototype response. Additionally, as was pointed out in section 3.5, since the number of feedback variables now equals the degree of the characteristic equation, we can rewrite the A_i in equation (28) as

$$A_i = B_{0i} K a_0 + B_{1i} K a_1 + \ldots\ldots\ldots + B_{ni} K a_n + B_n \qquad ,$$

thereby providing the designer with yet another potential variable.

To demonstrate the use of this algebraic design technique, consider again the third order example of figure (13). Let the open loop gain be variable, K, instead of 1.0 as shown. Furthermore, let the coefficient of the position feedback path be a_0. Then, for minimum prototype response, the characteristic equation becomes

$$z^3 + A_2 z^2 + A_1 z + A_0 = 0$$

where $A_2 = 0.2325Ka_2 + 0.1998Ka_1 + 0.0840Ka_0 - 1.503 = 0$

$$A_1 = -0.4650Ka_2 - 0.1263Ka_1 + 0.1709Ka_0 + 0.5529 = 0$$

$$A_0 = 0.2325Ka_2 - 0.0734Ka_1 + 0.0188Ka_0 - 0.0498 = 0$$

The simultaneous solution of these equations then, is

$$Ka_0 = 3.652$$

$$Ka_1 = 4.448$$

$$Ka_2 = 1.323$$

In order to permit the adoption of the conventional error definition, then,

$$K = 3.652$$

$$a_0 = 1.0$$

$$a_1 = 1.218$$

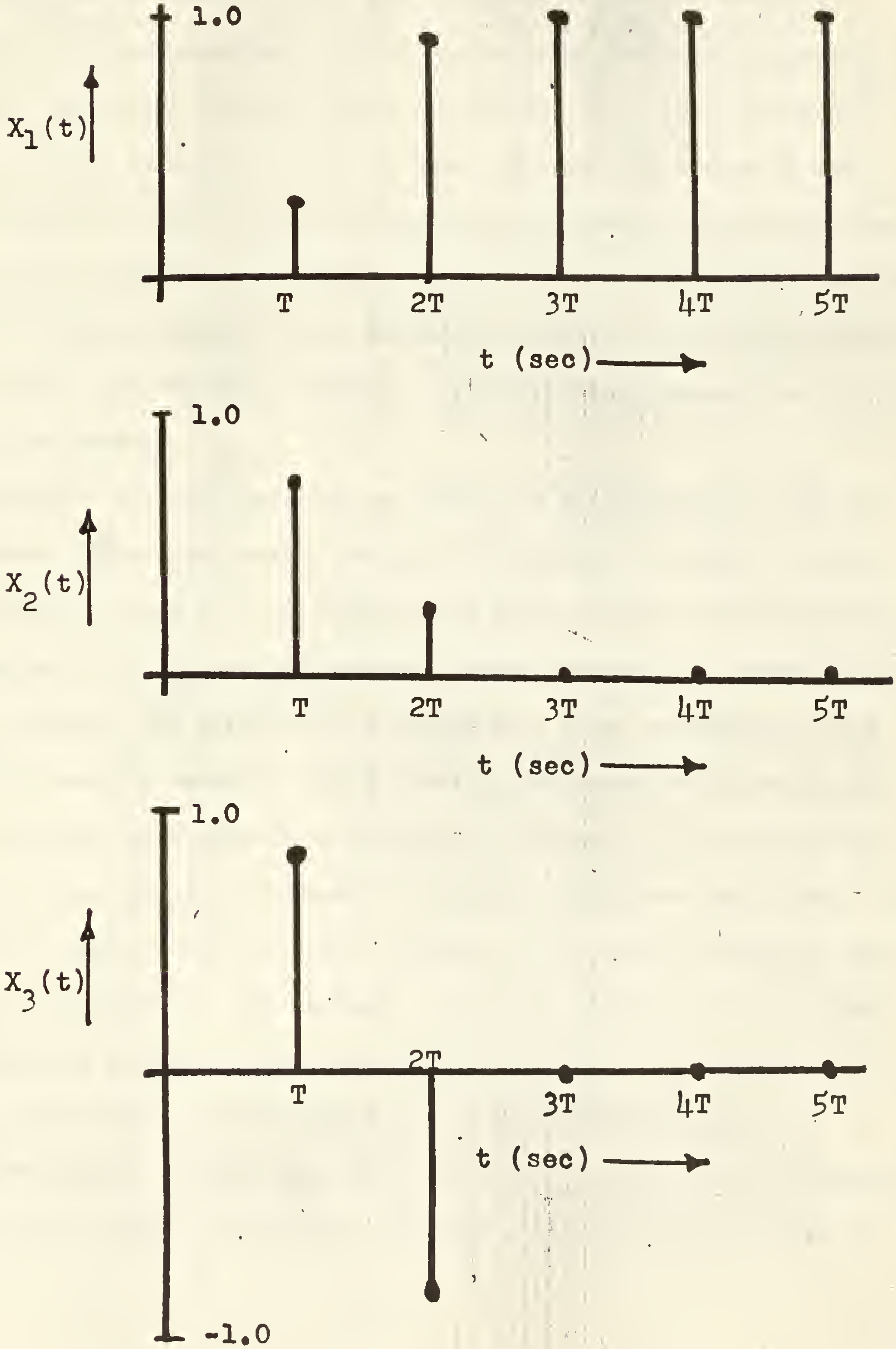
$$a_2 = 0.363$$

Using these values of the system parameters, the response of the system of figure (10) to a unit step input is as shown by table 6. The time response of the state variables is shown on figure (17).

TABLE 6

nT	$X_1(t)$	$X_2(t)$	$X_3(t)$
0	0.000	0.000	0.000
1	0.307	0.730	0.849
2	0.929	0.286	-0.849
3	1.000	0.000	0.000
4	1.000	0.000	0.000

Figure 17
State Variable Time Response for Algebraic Compensated Third Order System



CHAPTER IV

DIGITAL COMPUTATION IN MULTI-ORDERED PARAMETER SPACE

While the feedback compensation of many n-th order sampled-data control systems can be reduced to the two parameter problem presented in Chapter III, circumstances can, and do, arise where derivative signals higher than the second derivative are available. Since each additional degree of freedom available to the designer enlarges the region in the z-plane in which the closed loop roots may be positioned, common-sense alone dictates their usage where available. As pointed out in preceding chapters, Mitrovic's Method is primarily two parameter in nature. Additional design techniques are then desirable whereby a parameter space higher than a 2-space can be achieved.

The digital computer provides one method of increasing the order of the parameter space. Any design process or technique is simply a method for locating the roots of a polynomial, the system characteristic equation, as a function of one or more parameters. These methods were developed primarily to surmount the difficulty of factoring a given polynomial. With the digital computer however, this polynomial factoring can be repeatedly performed, always with greater accuracy, and sometimes in less total time than analysis performed by the graphical and semi-graphical techniques. In this chapter, one approach to multi-parameter study using the digital computer will be presented. The emphasis on the application of this method will be directed toward feedback compensation; however, the extension to cascade compensation or other parameter variations does exist.

Recent emphasis on the state space method of analysis, [2], as applied to sampled-data systems affords an excellent, if not necessary, point of

departure for digital computer analysis. The general form of the state space vector-matrix equation describing a sampled-data feedback control system was given by equation (14),

$$\underline{X}\,[(k+1)T] = \Phi\,\underline{X}(kT) + \Delta\,R(kT) \quad .$$

It can easily be demonstrated, see appendix 1, that the characteristic equation of the state transition matrix, Φ, is indeed the characteristic equation of the closed-loop transfer function. Since numerous numerical methods presently exist for determining the eigenvalues of a given matrix, the determination of the closed loop pole locations by evaluating the eigenvalues of the Φ matrix is readily performed on the digital computer. To determine these root locations as a function of one or more parameters, one has only to repeatedly increment the parameters and determine the eigenvalues of the resulting Φ matrix. In this fashion, an n-coordinate space can readily be created by computer programming techniques. One method is to use the FORTRAN language, and program DO LOOP's inside other DO LOOP's until sufficient coordinates are defined to permit the designer to sweep through a desired region in the n-space, determining the root locations corresponding to each point in the n dimensional parameter space.

In order to demonstrate this method of analysis more fully, the following specific project was undertaken. Given the perfectly general second order sampled-data system shown in figure (18), the values of the parameters Ka_1 and Ka_2 required for minimum prototype, minimum settling time response to a step input are to be determined as functions of the sampling interval, T. The resulting computer program should be such that the desired design data can be readily generated with only a minimum amount of additional input data supplied to the data, for any second order system with real poles.

The system of figure (18) can be represented by the open-loop vector matrix equation:

$$\underline{X}[(k+1)T] = F\underline{X}(kT) + \underline{\Delta}\, e(kT).$$

The error signal at the k-th sampling interval is represented as;

$$e(kT) = R(kT) - a_1X_1(kT) - a_2X_2(kT) \quad .$$

In the chosen example, the F matrix is perfectly general, irregardless of the location of the plant poles and zeros, and

$$F = Q_0 \begin{bmatrix} p_2e^{-p_1T} - p_1e^{-p_2T} & e^{-p_1T} - e^{-p_2T} \\ \\ Q_1(e^{-(p_1+p_2)T} - 1)/Q_0 & -p_1e^{-p_1T} + p_2e^{-p_2T} \end{bmatrix}$$

where

$$Q_0 \triangleq 1/(p_2 - p_1)$$

$$Q_1 \triangleq p_1p_2/(p_1 + p_2)$$

Modifications to the F matrix are needed, however, if a pure second order system is used. The $\underline{\Delta}$ matrix on the other hand, is dependent upon the number of pure integrations in the forward path of the plant transfer function. For example, for $p_1 = p_2 = 0$,

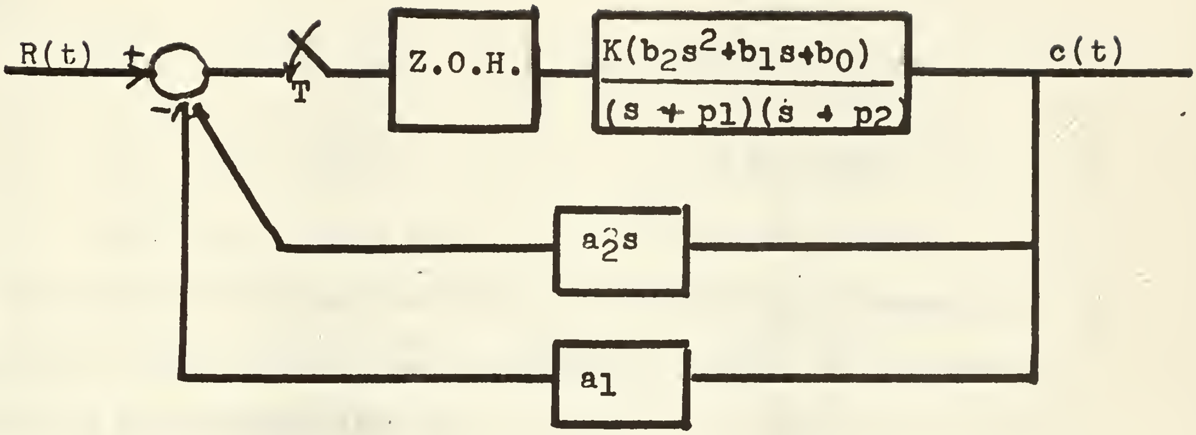
$$\underline{\Delta} = K \begin{bmatrix} b_2 + b_1T + b_0T^2/2 \\ \\ b_2\delta(T) + b_1 + b_0T \end{bmatrix}$$

When only one pure integration exists, that is $p_1 \neq 0$, $p_2 = 0$,

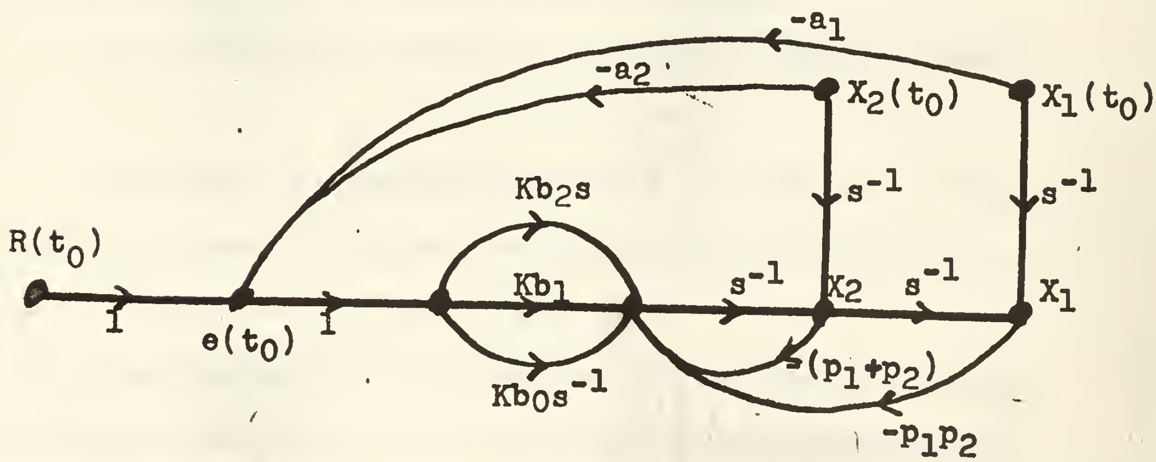
$$\underline{\Delta} = K \begin{bmatrix} (1/p_1)^2\,[(b_1 + b_0T)p_1 - b_0 + Q_2e^{-p_1T}] \\ b_2\delta(T) + b_o/p_1 - Q_2e^{-p_1T}/p_1 \end{bmatrix} ,$$

where,

$$Q_2 \triangleq (b_2p_1 - b_1)p_1 + b_0 \quad .$$



Block Diagram



Signal Flow Graph

Figure 18
General Second Order
System With Zeros

Finally, when $p_1 \neq 0$, and $p_2 \neq 0$,

$$\Delta = K \begin{bmatrix} b_0/p_1p_2 + Q_2Q_3e^{-p_1T} + Q_4Q_5e^{-p_2T} \\ \\ b_2\delta(T) + G_1Q_1e^{-p_1T} - G_2Q_1e^{-p_2T} \end{bmatrix} ,$$

where

$$Q_3 \triangleq 1/(p_1^2 - p_1p_2) \qquad G_1 \triangleq p_1^2 - p_1b_1 + b_0$$

$$Q_4 \triangleq 1/(p_2^2 - p_1p_2) \qquad G_2 \triangleq p_2^2 - p_2b_1 + b_0$$

$$Q_5 \triangleq (b_2p_2 - b_1)p_2 + b_0 \qquad \delta(T) = \text{unit impulse.}$$

The program which was developed, see appendix 2, to satisfy the design goal requires four inputs in the form of data. In the order in which they are required, they are:

a. the pole locations of the plant in the s-plane. The negative value of this s-plane location must be entered on the data card, and the poles should be listed in decreasing magnitude order.

b. the plant s-plane numerator coefficients listed in ascending order.

c. the minimum and maximum values of T, the sampling interval, and the incremental value of the sampling interval to be used in the investigation. T minimum must be greater than zero.

d. miscellaneous title information and graph scales required for graph output in accordance with an existing graph subroutine (DRAW).

The program outputs are:

a. a tabulated listing of the minimum settling time values of Ka_1 and Ka_2 determined for each value of T used in the investigation.

b. a graph plot of the minimum settling time value of Ka_1 vs. T.

c. a graph plot of the minimum settling time value of Ka_2 vs. T.

Let us now consider the geometric significance of this program. It has been demonstrated that for a given value of the sampling interval T, the origin of the z-plane maps into a single point on the parameter plane for a second order system. However, if a three dimensional right-handed coordinate set is created, whose axes are Ka_1, Ka_2, and T, the minimum settling time point corresponding to $\omega_z = 0$ will map into a curve in the 3-space. The plots of Ka_1 and Ka_2 required for minimum settling time response vs. T can be thought of as the projection of this curve in 3-space onto the Ka_1-T and Ka_2-T planes. Similiarly, any n-th order system, with m degrees of freedom, where $m \leq n$, can be investigated in a computer oriented m-space and displayed to the designer by a set of m projections on the a_i vs. T, $(i = 1, 2, \ldots, m)$, planes. If $m = n$, a set of projections on the Ka_i-T planes, $(i = 1, 2, \ldots, n)$, are available.

The following examples will serve to illustrate the class of feedback compensation problems, the solutions of which can be greatly simplified by using this particular programming technique.

EXAMPLE #1

A sampled-data control system whose plant transfer function is

$$G(s) = K(s + 1)/s(s + 2) \quad ,$$

is to be used with a dual capacity for sampling rates of 10cps and 1cps. Compensation is to be designed to provide minimum prototype response to a step input for each of the possible sampling frequencies.

Figures (19) and (20) are the minimum settling time parameter planes for this plant transfer function. From these parameter planes, the required parameter values for each sampling frequency can be obtained as:

T	Ka_1	Ka_2
0.1 sec	10.0	0.949
1.0 sec	1.15	0.538

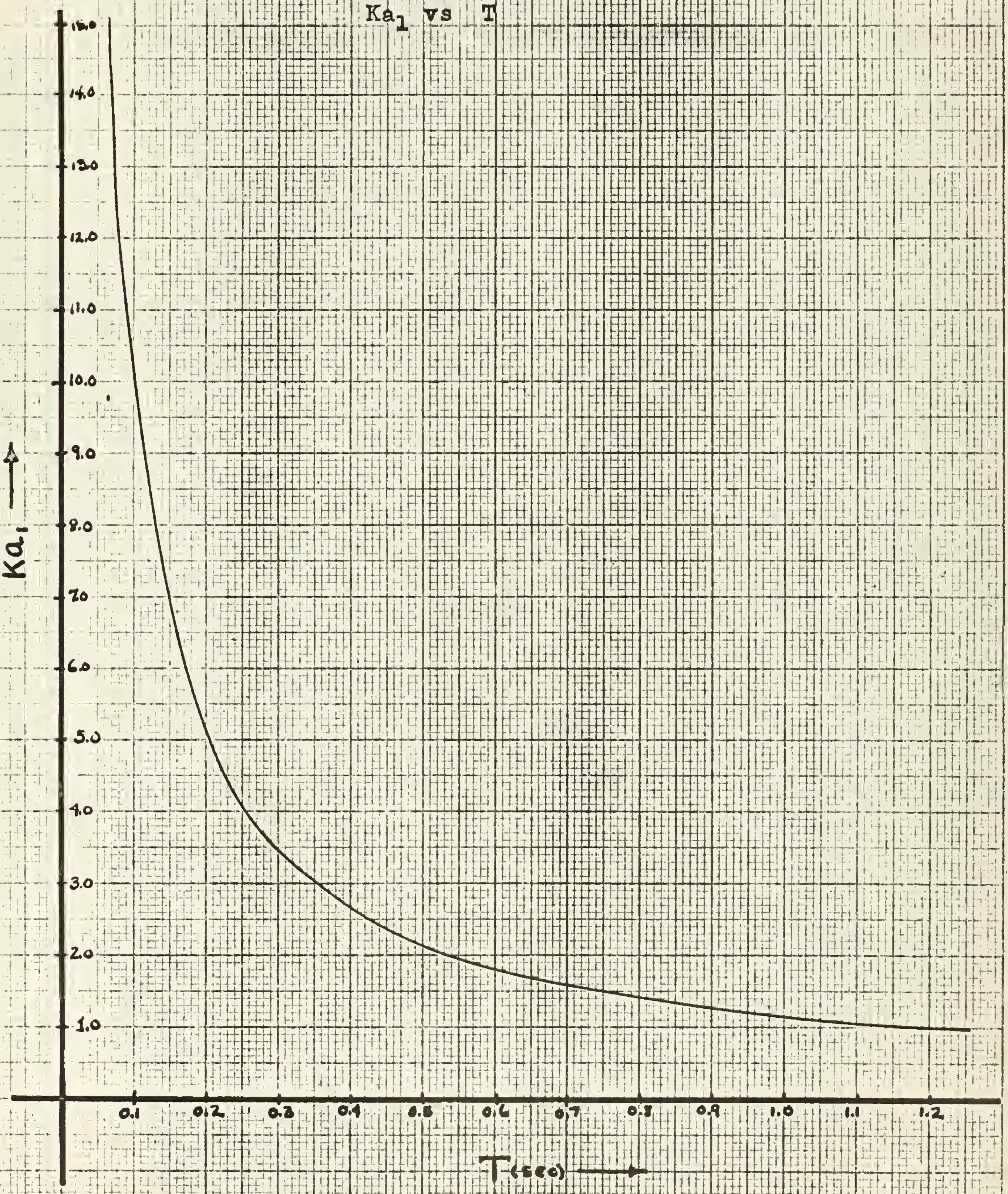
Finally, for unity position feedback, the physical plant parameters must be:

T	Ka_1	Ka_2
0.1 sec	10.0	0.095
1.0 sec	1.15	0.467

Figure (21) represents the block diagram of the compensated system. When the three switches in the physical plant are "ganged", then the controlled switch of the sampling frequency will introduce the correct, associated feedback compensation for minimum settling time.

Figure 19
Minimum Settling Time
Parameter Plane
Example 1

Ka_1 vs T



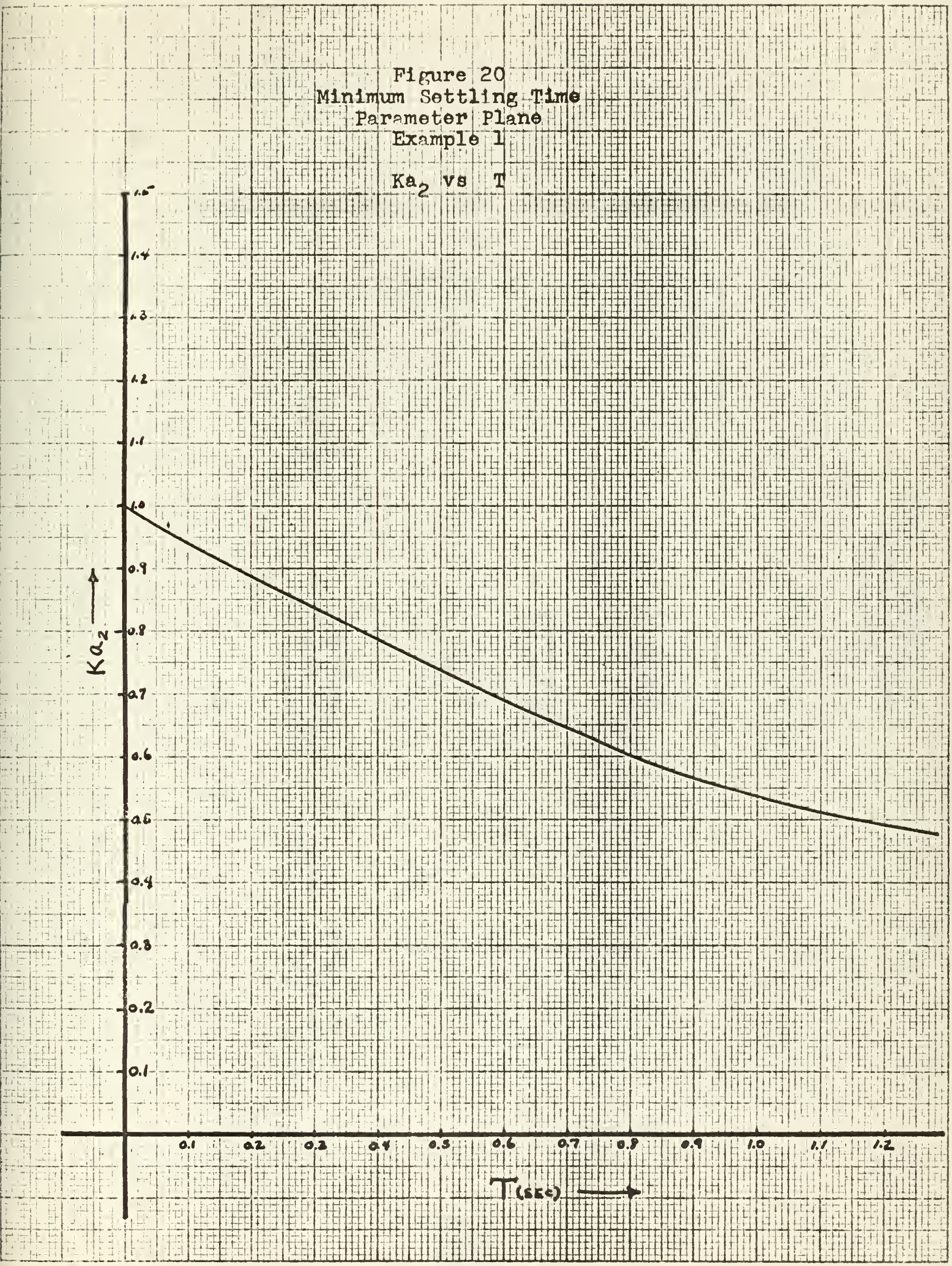


Figure 20
Minimum Settling Time
Parameter Plane
Example 1
Ka2 vs T
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
Ka2
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
T (sec)

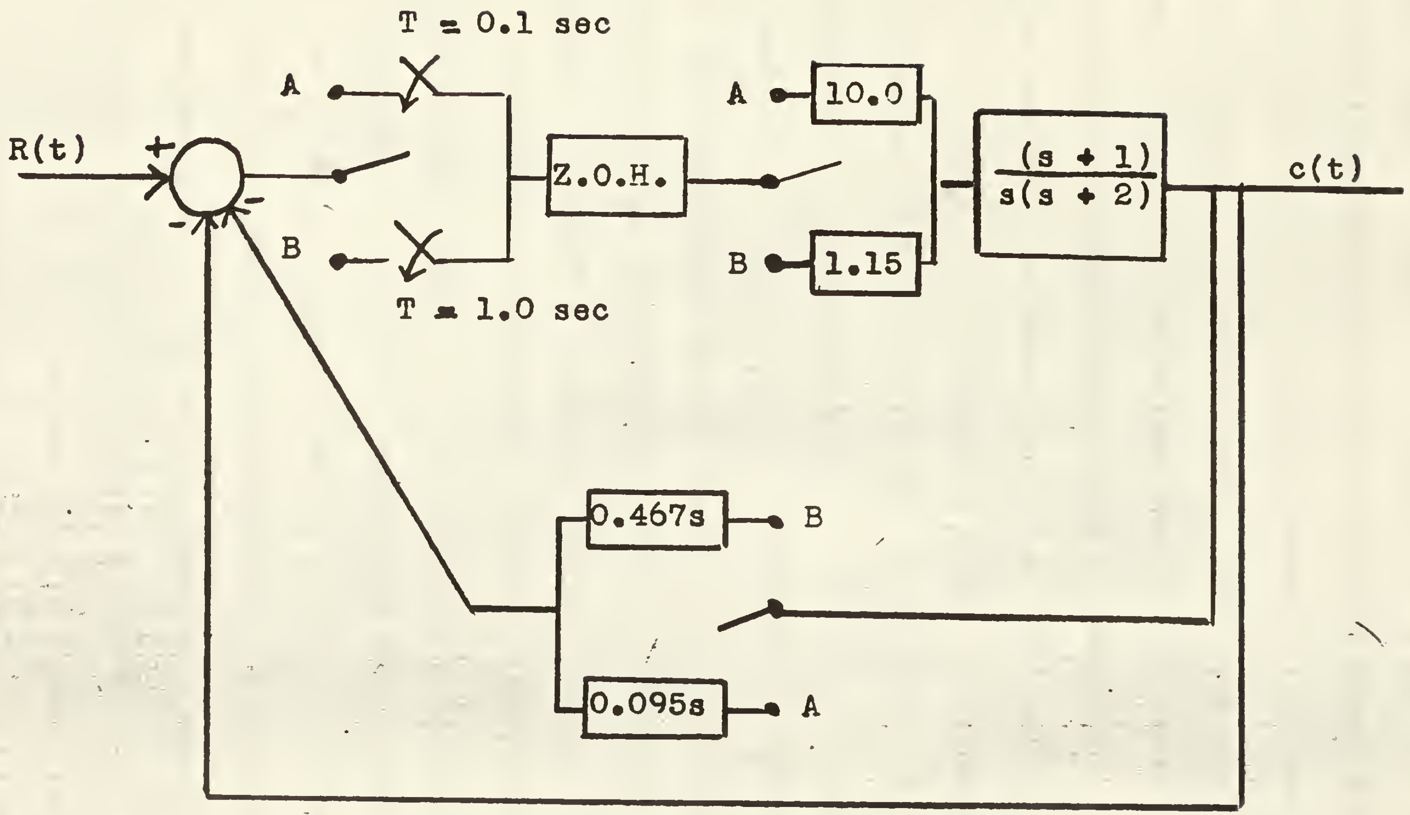


Figure 21
Block Diagram For
Dual Sampling Rate
Example

EXAMPLE #2

A system whose plant transfer function is

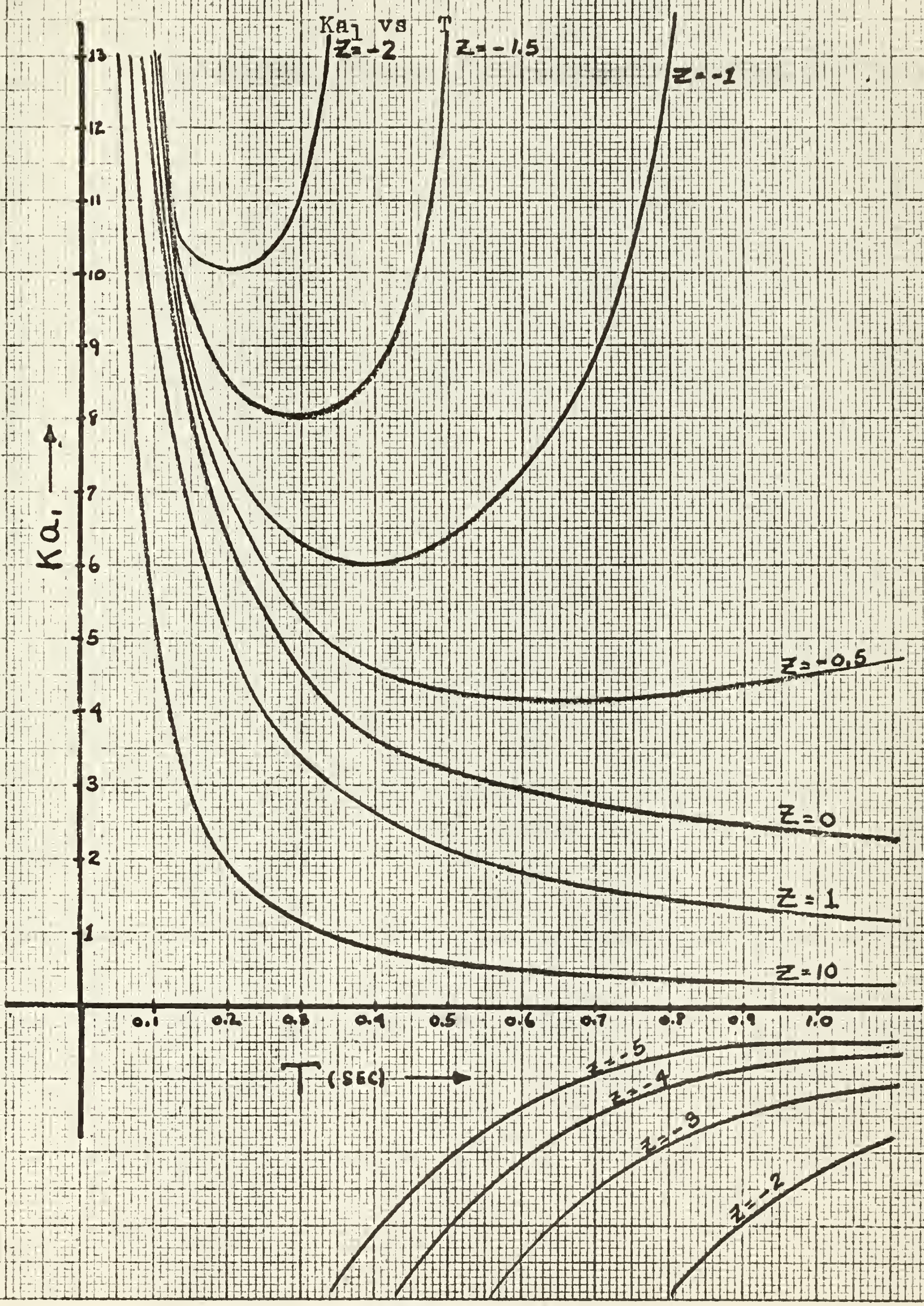
$$G(s) = K(s + Z)/s(s + 2)$$

is to be controlled by a sampled error signal. The plant zero location is a function of a cascaded compensator, and can be placed anywhere on the real axis of the s-plane. A sampling interval of 0.5 seconds is intended. The designer is to set the gain, K, the zero location, Z, and design feedback compensation for minimum settling time in response to a step input, and minimum steady state error for a ramp input.

Using the program to determine the minimum settling time values of Ka_1 and Ka_2 as a function of T, a family of curves, with parameter Z, can be constructed on the Ka_1 vs. T and Ka_2 vs. T planes. These families of curves are shown by figures (22) and (23). Of these two families of curves, the minimum settling time Ka_1 vs. T plane is by far the most interesting. The minimum settling time Ka_1 vs. T loci was shown by figure (19) to be hyperbolic. Furthermore, section 3.3 outlined the presence of discontinuities on parameter planes for systems with more than two parameters. These discontinuities again appear on the minimum settling time parameter planes, figures (22) and (23), and are best demonstrated by figure (23). Inspection of figure (22), the minimum settling time Ka_1 vs. T plane, also reveals these discontinuities on the loci of minimum settling time for the cases of Z less than zero (right-half s-plane zeros).

It has been demonstrated, [2] and [7], that for a type one sampled-data control system, the steady state error in response to a ramp input is inversely proportional to the plant root locus gain; similiar to the "velocity lag" error in continuous systems. Consequently, to minimize the ramp error and at the same time design for minimum settling time response to a

Figure 22
Minimum Settling Time
Parameter Plane
Example 2



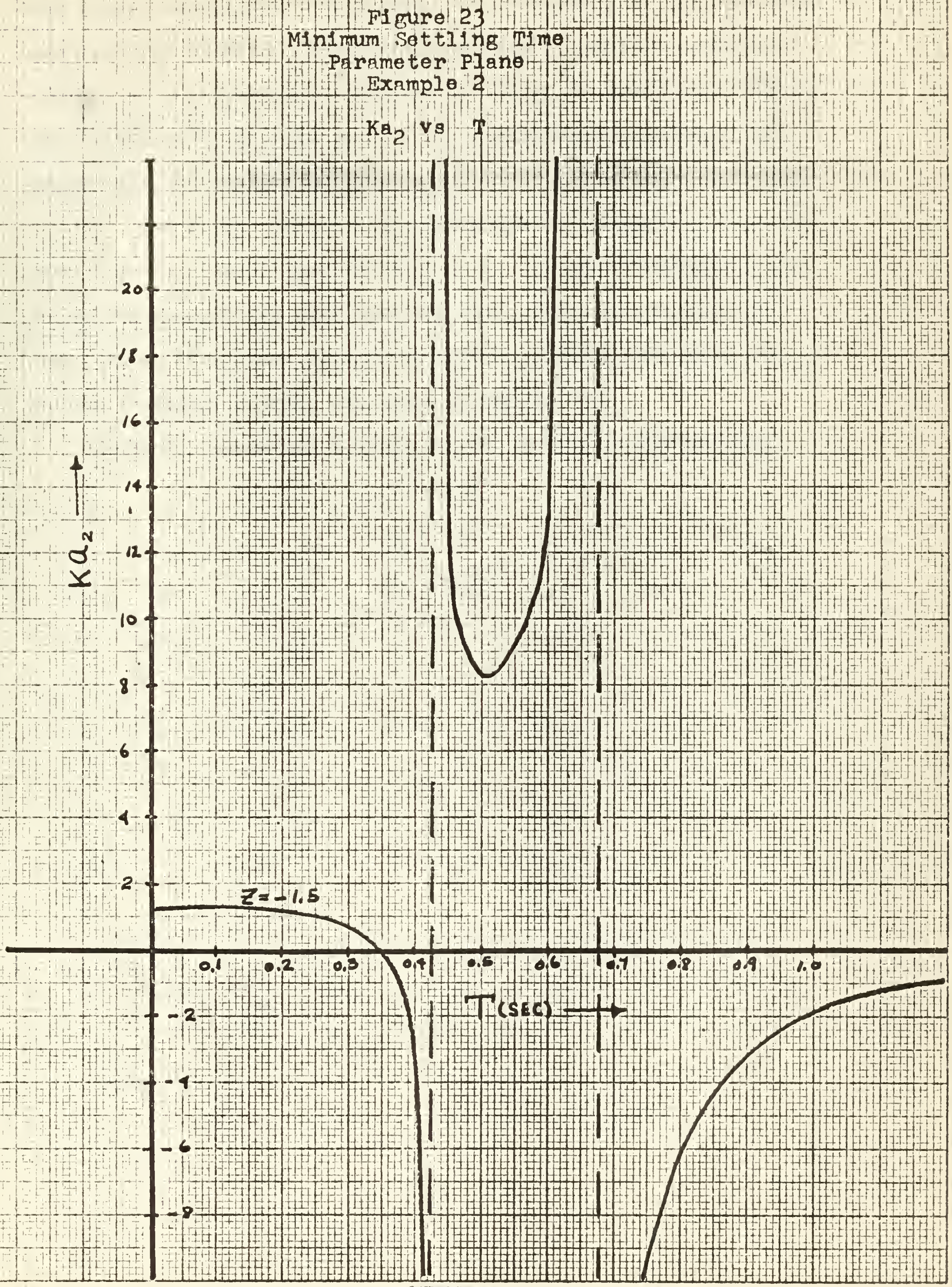


Figure 23
Minimum Settling Time
Parameter Plane
Example 2
Ka2 vs T
Ka2
T (SEC)
Z = -1.5
20
18
16
14
12
10
8
6
4
2
-2
-4
-6
-8
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

step input, the designer must maximize the parameter K (or Ka_1 when using unity position feedback) on the specified line of constant sampling interval T = 0.5. Inspection of figure (22) shows the importance played in this selection of the plant gain by the discontinuity phenomena. From figure (22), the maximum Ka_1 attainable for T = 0.5 is Ka_1 = 12.7, when Z =-1.5. For Z less than -1.5, the discontinuity asymptote lies to the left of T = 0.5, and a negative value of Ka_1 (positive feedback) is required for minimum step input settling time. For Z greater than -1.5, lower values of Ka_1 are required for minimum settling time for a step input and therefore a larger ramp error is experienced.

Therefore, the design criteria of this example is achieved for:

$$K = 12.7$$

$$a_1 = 1.0$$

$$a_2 = 0.656$$

$$Z = -1.5$$

CHAPTER V

CONCLUSIONS

The application of Mitrovic's Method to feedback compensation of sampled-data control systems has been shown to be quite useful in the minimization of system settling time. The parameter plane for the second order system presents an easily obtainable picture of the system stability and settling time as a function of the system parameters. Using the methods described herein, the second order parameter plane development results in a two parameter analysis method requiring about the same total effort to construct as conventional one parameter methods.

In the case of higher order systems, the effectiveness of the Mitrovic Method as a design tool is somewhat reduced. Foremost, the plotting of the various loci requires at least a desk calculator, preferably a digital computer. If the number of system parameters can be reduced to only two, then the parameter plane is still a good tool for minimizing settling time. Unfortunately, the determination of the minimum settling time compensation from the higher order parameter plane requires a more tedious study of the plane than for the second order case. The intersections of all the loci of constant ω_z and constant σ_z must be examined in detail in search of the circle of minimum radius in the z-plane encircling all the closed loop roots.

For higher order systems with more than two degrees of freedom, analysis with the digital computer is most effective, indeed necessary. The vector-matrix state space equations representing the given system were shown to lead directly to the determination of the closed loop root positions as a function of up to n parameters, n being the order of the

system. The strength of this method is the ability to reduce the design criteria to a function of almost any physical parameter or condition. However, since a large fraction of the feedback compensation problems are not more than two parameter in nature, the Mitrovic Method can be an exceedingly important tool to the design engineer.

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APPENDIX I

STABILITY ANALYSIS OF VECTOR-MATRIX DIFFERENCE EQUATIONS

The general form of the state space vector-matrix equation describing a linear system with a sampled input is,

$$\underline{X}[(k+1)T] = F\underline{X}(kT) + \underline{\Delta}\, u(kT). \tag{I-1}$$

In this form, the states of the system, $\underline{X}$, are defined as the control system output and its first n-1 derivatives, where n is the order of the linear system. For a feedback control system, the control vector, u(t), is the difference between the command input, R(t), and the sum of the feedback quantities, ie;

$$u(t) = R(t) - \underline{A}^t\, \underline{X}(t), \tag{I-2}$$

where $\underline{A}^t$ is the transpose of $\underline{A}$, the coefficient vector controlling the magnitude of feedback for each state variable. In discrete time form, equation (I-2) is written as

$$u(kT) = R(kT) - \underline{A}^t\, \underline{X}(kT) \quad . \tag{I-3}$$

By substituting equation (I-3) into equation (I-1), the familiar vector-matrix difference equation for a sampled data control system is obtained as

$$\underline{X}[(k+1)T] = F\underline{X}(kT) - \underline{\Delta}\,\underline{A}^t\, \underline{X}(kT) + \underline{\Delta}\, R(kT) \quad , \tag{I-4}$$

or by imbeding the feedback quantities directly into the F matrix,

$$\underline{X}[(k+1)T] = \Phi\, \underline{X}(kT) + \underline{\Delta}\, R(kT) \quad . \tag{I-5}$$

With the feedback thus imbeded in the transition matrix, Φ, the stability analysis of a given system can be confined to an analysis of the partial state equation,

$$\underline{X}[(k+1)T] = \Phi\, \underline{X}(kT) \quad , \tag{I-6}$$

which is an initial value problem. If the state trajectory returns to a

given equilibrium, or singular point, in the state space for any initial state vector, it will also move to a predictable translation of this singular point and achieve equilibrium for the input R(t) which can be described by the initial state vector. The dynamic stability of the system is thus ensured.

Hahn, [1], studied the stability of equation (I-6) using Liapunov's Direct Method, and concluded that the stability of equation (I-6) was ensured when the logarithm of the eigenvalues of the Φ matrix were all less than zero in magnitude, that is

$$\ln \lambda_i < 0 \quad . \tag{I-7}$$

This is equivalent to

$$\ln (\mathrm{Re}\,\lambda_i + j\ \mathrm{Im}\,\lambda_i) < 0$$

or

$$\ln (|\lambda_i|\ e^{j\phi}) < 0$$

or

$$\ln |\lambda_i| + j (\phi \pm 2n\pi) < 0$$

which is satisfied if and only if

$$|\lambda_i| < 1 \quad .$$

Immediately, a direct analogy of the unit circle in the λ - plane can be drawn to the unit circle of the z-plane. This analogy can also be demonstrated by considering the z-transformation of equation (I-6) as:

$$z\underline{X}(z) = \Phi\ \underline{X}(z) + z\underline{X}(0)$$

$$(zI - \Phi)\underline{X}(z) = z\underline{X}(0)$$

or,

$$\underline{X}(z) = z(zI - \Phi)^{-1}\ \underline{X}(0)$$

Now, the inverse of the matrix $(zI - \Phi)$ is the transpose of the cofactor matrix, divided by the determinant value of $(zI - \Phi)$; or, if the cofactor transpose matrix is denoted as [C]

$$(zI - \Phi)^{-1} = \frac{[C]}{|zI - \Phi|}$$

Then

$$\underline{X}(z) = \frac{z[C]}{|zI - \Phi|} \underline{X}(0)$$

Since z is just a scalar variable over a complex region, the determinant $|zI - \Phi|$ is nothing more than the characteristic equation of the matrix, whose characteristic values or eigenvalues are the closed loop root locations in the z-plane, or λ -plane. Similiarly, in equation (I-1), the eigenvalues of the F matrix are the open loop root locations in the z-plane (λ -plane). The vector-matrix formulation is ideally suited for digital computation of the state variable time response in discrete form. Even more important, it is also suited for determination of the system dynamic stability by digitally locating the eigenvalues of the Φ matrix as a function of any variable in the Φ matrix.

APPENDIX II

FORTRAN PROGRAM FOR MINIMUM SETTLING TIME PARAMETER PLANE

```
      PROGRAM DESIGN
C     PROGRAMMER L.D.NACE
C     THIS PROGRAM COMPUTES AND GRAPHS THE VALUES OF THE
C     PARAMETERS KA1 AND KA2 REQUIRED FOR MINIMUM PROTOTYPE
C     RESPONSE OF ANY SECOND ORDER SAMPLED-DATA FEEDBACK
C     CONTROL SYSTEM AS A FUNCTION OF THE SAMPLING INTERVAL T.
C     INPUT DATA IS
C     CARD 1  VALUES OF P, THE NEGATIVE OF THE PLANT S-PLANE
C     POLE LOCATION IN DECREASING MAGNITUDE ORDER, FORMAT 2F10.5
C     CARD 2  COEFFICIENTS OF THE S-PLANE NUMERATOR OF PLANT
C     IN ASCENDING ORDER, FORMAT 3F10.5
C     CARD 3  T MIN, T MAX, DELTA T, IN FORMAT 3F10.5. T MIN IS
C     THE MINIMUM VALUE OF T SCANNED. T MAX IS THE MAXIMUM
C     VALUE, AND DELTA T IS THE INCREMENTAL VALUE OF T USED IN
C     CALCULATIONS. MAXIMUM NUMBER OF POINTS COMPUTED IS 500.
C     T MIN MUST BE GREATER THAN ZERO.
C     CARD 4  FIRST LINE OF THE GRAPH TITLE, COLUMNS 1-48.
C     CARD 5  SECOND LINE OF THE GRAPH TITLE, COLUMNS 1-48
C     CARD 6   EXSCALE,YSCALE FOR GRAPH OUTPUT IN FORMAT 2F10.5
      DIMENSION P(2),B(3),ANS1(500),ANS2(500),TT(500)
      DIMENSION ITITLE(12)
      READ 100,P
  100 FORMAT (2F10.5)
   75 FORMAT (/////)
   76 FORMAT (//)
      PRINT 75
      PRINT 101
  101 FORMAT (47HVALUES OF P, NEGATIVE OF S-PLANE POLE LOCATIONS)
      PRINT 76
      PRINT 100,(P(I),I=1,2)
      READ 102,B
  102 FORMAT (3F10.5)
      PRINT 75
      PRINT 103
  103 FORMAT (47HPLANT NUMERATOR COEFFICIENTS IN ASCENDING ORDER)
      PRINT 76
      PRINT 102,(B(I),I=1,3)
      READ 102, TMIN,TMAX,DELTAT
      PRINT 104
  104 FORMAT (/////,3X,5HT MIN,6X,5HT MAX,6X,7HDELTA T,//)
      PRINT 102, TMIN,TMAX,DELTAT
  200 FORMAT (6A8)
      READ 200, (ITITLE(I),I=1,6)
      READ 200,(ITITLE(I),I=7,12)
      PRINT 201
  201 FORMAT (/////,12HGRAPH TITLES,//)
```

```
      PRINT 200,(ITITLE(I),I=1,6)
      PRINT 200,(ITITLE(I),I=7,12)
      READ 202, EXSCALE,YSCALE
  202 FORMAT (2F10.5)
      PRINT 203
  203 FORMAT (/////,10X,7HX-SCALE,10X,7HY-SCALE,//)
      PRINT 204,EXSCALE,YSCALE
  204 FORMAT (8X,F10.5,9X,F10.5)
      LAST = 0
      P1=P(1)
      P2=P(2)
      B0=B(1)
      B1=B(2)
      B2=B(3)
      K=0
      T=TMIN
  105 K=K+1
      IF (P1+P2)    118,112,118
  118 U=1./EXPF(P1*T)
      V=1./EXPF(P2*T)
      PH11=P2*U/(P2-P1)+P1*V/(P1-P2)
      PH12=U/(P2-P1)+V/(P1-P2)
      PH21=(U*V-1.)*((P1*P2)/(P1+P2))
      PH22=P1*U/(P1-P2)+P2*V/(P2-P1)
      N=0
      IF (P1)    107,106,107
  106 N=N+1
  107  IF  (P2)    109,108,109
  108 N=N+1
  109 IF (N-1)    110,111,112
  110 D1=B0/(P1*P2)+((B2*P1*P1-B1*P1+B0)/(P1*P1-P1*P2))*U
     1+((B2*P2*P2-B1*P2+B0)/(P2*P2-P1*P2))*V
      D2=((B1-P1-P2)*(-P1)+B0-P1*P2)*U/(P2-P1)
     1+((B1-P1-P2)*(-P2)+B0-P1*P2)*V/(P1-P2)
      GO TO 113
  111 D1=((B1+B0*T)*P1-B0)/(P1*P1)+((B2*P1*P1-B1*P1+B0)*U)/(P1*P1)
      D2=B0/P1-((B2*P1-B1)*P1+B0)*U/P1
      GO TO 113
  112 D1=B2+B1*T+B0*T*T/2.
      D2=B1+B0*T
      PH11=1.
      PH12=T
      PH21=0.
      PH22=1.
  113 DENOM=D1*(PH21*D1-PH11*D2)-D2*(PH12*D2-PH22*D1)
      DUMA1=(PH11+PH22)*(PH21*D1-PH11*D2)
     1+D2*(PH11*PH22-PH12*PH21)
      DUMA2=-D1*(PH11*PH22-PH12*PH21)-(PH11+PH22)*(PH12*D2-PH22*D1)
      ANS1(K)=DUMA1/DENOM
      ANS2(K)=DUMA2/DENOM
      TT(K)=T
      IF (TMAX-T)    115,115,114
  114 T=T+DELTAT
      GO TO 105
```

```
115 PRINT 116
116 FORMAT (/////,10X,1HT,19X,3HKA1,17X,3HKA2,//)
    PRINT 117,  (TT(J),ANS1(J),ANS2(J),J=1,K)
117 FORMAT(5X,F10.5,10X,F10.5,10X,F10.5)
    CALL DRAW (K,TT,ANS1,0,0,4HKA1 ,ITITLE,EXSCALE,YSCALE,
   10,0,0,0,8,8,1,LAST)
    CALL DRAW (K,TT,ANS2,0,0,4HKA2 ,ITITLE,EXSCALE,YSCALE,
   10,0,0,0,8,8,1,LAST)
    END
    END
```